AD 272 908

Reproduced by the

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

REPORT NO.
TDR-930(2230-09)TN-2

The Status of Unsteady Aerodynamic Influence Coefficients

22 NOVEMBER 1961

Prepared by WILLIAM P. RODDEN, *Aerospace Corporation*
JAMES D. REVELL, *Northrop Corporation*

Prepared for DEPUTY COMMANDER AEROSPACE SYSTEMS
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
Inglewood, California

AEROSPACE CORPORATION
CONTRACT NO. AF 04(647)-930

Report No.
TDR-930(2230-09)TN-2

THE STATUS OF UNSTEADY AERODYNAMIC INFLUENCE COEFFICIENTS

by

William P. Rodden
Aerospace Corporation

and

James D. Revell
Norair Division, Northrop Corporation

AEROSPACE CORPORATION
El Segundo, California

Contract No. AF 04(647)-930

22 November 1961

Prepared for
DEPUTY COMMANDER AEROSPACE SYSTEMS
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
Inglewood, California

ABSTRACT

The most general numerical formulation that can be made of the aerodynamic lifting problem is a matrix of aerodynamic influence coefficients (AICs) that relate the control (collocation) point forces (the local surface integral of the pressure) to the control point downwash velocities (or the control point deflections, whose substantial derivatives are the downwash velocities). The AICs are necessary in any aeroelastic analysis by collocation methods. Furthermore, the AICs offer a <u>large</u> computational advantage in aeroelastic analyses by modal methods since the AICs need be derived <u>only</u> <u>once</u> and the generalized forces follow from multiplications with the modal matrix; any approach that derives generalized forces directly is inefficient to the extent that the aerodynamic analysis must be repeated for <u>every</u> revision of the modal set.

Downwash is considered to arise from either motion or a gust, and the unsteadiness is considered to be oscillatory, transient, or static (a limit of the oscillatory case). The oscillatory AICs are defined by

$$\left\{F\right\} = \rho\omega^2 b_r^2 s \left[C_h\right]\left\{h\right\}$$

for motion, and by

$$\left\{F_g\right\} = \rho V W_g b_r s \left[C_g\right]\left\{w_g/W_g\right\}$$

for a gust. The static AICs are defined by

$$\left\{F_s\right\} = (qS/\bar{c})\left[C_{hs}\right]\left\{h\right\}$$

The use of AICs thus defined is illustrated in collocation formulations of several aeroelastic problems, and the use of the AICs in analyses by modal methods is discussed. Several possibilities for the definition of transient AICs are suggested in a preliminary review of the transient problem.

The status of AICs is discussed for surfaces of various planforms by two-dimensional (i.e., strip) theories and by three-dimensional theories, for slender bodies and for panels, throughout the range of Mach number of current interest. A unified lifting surface theory approach to AICs is demonstrated in the subsonic-transonic-supersonic Mach number regimes based on a critical review of current techniques. A systematic approach to interference effects is suggested. A critical assessment of transonic and hypersonic problems is presented, and the linear approximations of slender-wing theory and piston theory are discussed in detail to illustrate the AIC development. A result is derived for the AICs from slender body theory, and the significance of first- and second-order slender body theory is reviewed. The AICs for a two-dimensional panel are given, and the problem of adapting three-dimensional aerodynamic theory to the finite panel problem is mentioned. The oscillatory AICs for a harmonic gust are developed from incompressible strip theory, and a method of extension of the AICs for motion to obtain the gust AICs from lifting surface theory is shown. A discussion is given of current investigations of possible formulations of transient AICs and of related computational problems. An empirical weighting matrix that permits adjustment of the theoretical AICs to incorporate the results of experimental measurements is suggested, and, finally, a number of recommendations for further investigations and experimental correlation are made.

CONTENTS

CONTENTS (continued)

FIGURES

SYMBOLS

A^i_{nm}	Element of matrix relating pressure loading function amplitude a_{nm} to downwash at point i
a	Element of flexibility matrix; speed of sound; panel dimension
a_{mn}	Amplitude of deflection mode
a_{nm}	Amplitude of pressure loading function
B^{ij}_{mn}	Element of integration matrix for slender-wing theory
B^j_{nm}	Element of integration matrix relating control point forces to pressure loading function amplitudes
b	Local semichord; constant in exponential approximation of aerodynamic lag; panel dimension
b_r, b_o	Reference semichord, and root semichord, respectively
C_F, C_{Fi}	Skin friction coefficient for compressible and incompressible flows, respectively
C_g	Element of AIC matrix for a harmonic gust
C_h	Element of oscillatory AIC matrix for motion
C_{hs}	Element of static AIC matrix
C_p	Pressure coefficient
C_Z	Lift coefficient; C_{Z_α} is lift curve slope
c	Root chord; panel span; c_a is control surface chord; $\bar{c}$ is mean aerodynamic chord
D^i_{nm}	Element of matrix relating downwash at point i to pressure loading function amplitude a_{nm}
d	Distance between forward and middle control points; diameter

d_{rs}	Amplitude of deflection mode
F	Control point force
$F_{\nu N}/\epsilon_N$	Downwash factor for concentrated chordwise replacement load
$G_{\nu n}$	Element of matrix relating ΔP_ν to a_{nm}
g	Structural damping coefficient
$g_m(\eta)$	Spanwise loading function
H	Boundary layer parameter, $H = \delta^*/\theta$
H_{mn}, H_r, H_{rs}	Assumed deflection functions
h	Surface deflection
I	Element of unit matrix; momentum of cross-flow virtual mass
I_n, J_n	Slender body spanwise integrals; piston theory thickness integrals
I_r^j	Element of deflection interpolation matrix
i	Imaginary unit
K	Kernel of downwash integral equation; constant; K_θ is nonsingular numerator of subsonic kernel function; K_{θ_b} and K are hypersonic similarity parameters
K_{mn}, L_{mn}	Slender-wing theory surface integrals
K_{mn}^{ij}	Element of slender-wing theory aerodynamic inertia matrix
k	Local reduced frequency, $k = b/V$
k_r	Reference reduced frequency
L	Lift
L_h, L_α, L_β	Oscillatory lift coefficients

ℓ	Surface semispan (in Section II.B.2, otherwise s is used)
$\ell_n(\theta)$	Chordwise pressure loading functions
M	Mach number; element of mass matrix; pitching moment
M_h, M_α, M_β	Oscillatory pitching moment coefficients
M_{mn}^{ij}	Element of slender-wing theory aerodynamic spring matrix
N_d	Number of downwash interference fields
N_{mn}^{ij}	Element of slender-wing theory aerodynamic damping matrix
n	Normal vector
ΔP_ν	Concentrated chordwise replacement load
p, Δp	Pressure
Q	Generalized force
q	Dynamic pressure; q_n is n'th generalized coordinate
Re	Reynolds number
R(x)	Body radius in cylindrical coordinates
r	Radial cylindrical coordinate
S	Surface reference area; body cross-sectional area
ΔS	Incremental area surrounding control point
s	Surface semispan (except in Section II.B.2 where ℓ is used)
T	Hinge moment; representative decay time; temperature, T_w is wall temperature, T_* is mean boundary layer temperature
T_h, T_α, T_β	Oscillatory hinge moment coefficients
t	Time
u	Streamwise velocity in boundary layer
V	Free stream velocity

W	Element of empirical weighting matrix
W_g	Reference downwash amplitude for harmonic gust
W_{ij}	Element of matrix relating downwash at points i to deflection points j
w	Downwash velocity
w_g	Local downwash amplitude for harmonic gust
x, y	Dimensional Cartesian coordinates; dimensionless Cartesian coordinates normalized to b_0, ℓ; (x_0, y_0) are dimensionless distances between source and downwash point; $(\bar{x}, \bar{y})$ are aerodynamic center coordinates
Δx	Incremental panel length
Δy	Strip width
α	Angle of attack
β	Control surface incidence; $\sqrt{\lvert 1 - M^2 \rvert}$; constant in exponential approximation of aerodynamic lag
γ	Ratio of specific heats
Δ_n	Incremental chordwise length
δ	Delta wing semiapex angle; body thickness parameter
$\delta'_{BL}, \delta^*, \delta_{SL}$	Boundary layer thickness, displacement thickness, and shock layer thickness, respectively
ϵ	Density ratio; thickness parameter; deviation in empirical weighting matrix analysis
θ	Chordwise angular coordinate; boundary layer momentum thickness; circumferential cylindrical coordinate; θ_b is body slope relative to free stream
Λ	Sweep angle: of quarter chord in subsonic analysis, of leading edge in supersonic analysis

Λ_0, Λ_1	Nonhomogeneous functions in second-order slender-body solution
λ	Complex eigenvalue, $\lambda = \lambda_R + i\lambda_I$; gust wave length
μ	Coefficient of viscosity
ξ, η	Dimensionless coordinates
ρ	Atmospheric density
σ	Ratio of semispan to root chord
τ	Dimensionless time; real time measured in inertial reference frame
Φ, ϕ	Potential functions
$\Phi(k)$	Sears incompressible harmonic gust function
χ	Hypersonic viscous interaction parameter; pressure potential function for flows with vorticity
ψ	Second-order potential function
Ω	Potential function
ω	Frequency

Subscripts and Supercripts

$(\ )_b$	Evaluation at body surface; refers to basic weighting factor
$(\ )^{(e)}$	Experimental
$(\ )^F$	Fuselage
$(\ )_f$	Evaluation in fluid
$(\ )_g$	Gust
$(\ )^i$	Index of downwash control point grid
$(\ )^j$	Index of deflection-load control point grid
$(\ )_{mn}$	Refers to deflection mode of amplitude a_{mn}

$(\ )_{nm}$	Refers to pressure loading coefficient a_{nm}
$(\ )_r$	Rigid; refers to redundant weighting factor
$(\ )_s$	Static; shock location
$(\ )^{(t)}$	Theoretical
$(\ )_x$	Partial derivative with respect to x; $(\ )_y$, $(\ )_z$, $(\ )_t$ denote differentiation with respect to y, z, and t, respectively
$(\ )^W$	Wing
$(\ )_0$	Free-stream condition; reference value; axial flow or symmetrical potential
$(\ )_1$	Cross flow or lifting potential; condition upstream of shockwave
$(\ )_2$	Condition downstream of shockwave
$\overline{(\ )}$	Complex amplitude
$(\ )'$	Derivative; disturbance value

<u>Matrix Notation</u>

$[\ ]$	Square, rectangular, row, or diagonal matrix
$\{\ \}$	Column matrix
$[\ ]^*$	Transposed matrix
$[\ ]^{-1}$	Inverse of matrix

I. INTRODUCTION

The essence of the aerodynamic lifting problem is the relationship between the distribution of lifting pressure (the pressure differential between the upper and lower surfaces) and the surface distribution of downwash velocity. The mathematical formulation of the lifting problem leads to an integral equation for the lifting pressure. The most general numerical formulation that can be made from the aerodynamic integral equation leads to a matrix of aerodynamic influence coefficients (AICs) that relate the control (collocation) point lifting pressures (or the control point forces, which are the local surface integrals of the pressures) to the control point downwash velocities (or the control point deflections, whose substantial derivatives are the downwash velocities). For aeroelastic analysis, AICs that lead to control point forces, rather than pressures, are more useful. In the problems arising from atmospheric disturbance (e.g., a gust), the AICs may relate the forces to the downwash directly; in the problems arising from motion of the surface (e.g., lift effectiveness, flutter), the AICs should relate the forces to the deflections or the streamwise slopes. In the present treatment, we prefer the deflections rather than streamwise slopes as the generalized coordinates, because the deflections have a more general meaning on a cambered surface, and deflection influence coefficients are more readily obtained from a structural analysis than slope influence coefficients. Accordingly, we have defined[1] a matrix of static AICs for quasi-static motion by

$$\left\{F_s\right\} = (qS/\bar{c}) \left[C_{hs}\right] \left\{h\right\} \quad . \tag{1}$$

We have defined[1,2] a (complex) matrix of oscillatory AICs for harmonic motion by

$$\left\{F\right\} = \rho \omega^2 b_r^2 s \left[C_h\right] \left\{h\right\} \quad . \tag{2}$$

We have defined[3] a (complex) matrix of oscillatory AICs for a harmonic gust by

$$\left\{F_g\right\} = \rho V W_g b_r s \left[C_g\right] \left\{w_g/W_g\right\} \quad . \tag{3}$$

We note that the definitions of Eqs. (1) to (3) are completely general, being equally applicable to two-dimensional (i. e., strip) or three-dimensional aerodynamic theories. We also note that the forces given by Eqs. (1) and (2) must be equal in the limit of zero frequency

$$\left\{F_s\right\} = \lim_{k_r \to 0} \left\{F\right\} \tag{4}$$

which leads to the relationship between the static AICs and the oscillatory AICs for motion

$$\left[C_{hs}\right] = \lim_{k_r \to 0} (2k_r^2\, s\bar{c}/S) \left[C_h\right] \tag{5}$$

Hence, the static AICs need no special consideration in the subsequent discussion. We do not wish to minimize the importance of the many developments in the static context.[4-12] However, we shall concentrate on the unsteady formulations because of the recent number of significant advances in unsteady aerodynamic theory, while recognizing that there are some problem areas, as we shall point out in our review of piston theory, that await the steady solution before the unsteady one is known.

We may demonstrate the general usefulness of AICs by considering the collocation formulation of a number of the aeroelastic problems. For the illustration, we choose the problems of static lift distribution, flutter, and the frequency response analysis required for structural fatigue analysis under random atmospheric turbulence. For simplicity, we assume no rigid body degrees of freedom. Our basic equation relates the structural deformations to the applied forces through the structural influence coefficients

$$\left\{h_f\right\} = [a] \left\{F_a\right\} \tag{6}$$

For the static aeroelastic analysis, the external forces are given by Eq. (1) where the deflection mode is found from the sum of the initial deflection mode of

the surface considered as rigid and the deformation mode

$$\left\{h\right\} = \left\{h_r\right\} + \left\{h_f\right\} \tag{7}$$

Substituting Eq. (6) into Eq. (7) and the result into Eq. (1) permits solution for the external loading:

$$\left\{F_a\right\} = [A] \left\{F_r\right\} \tag{8}$$

where

$$\left\{F_r\right\} = (qS/\bar{c}) [C_{hs}] \left\{h_r\right\} \tag{9}$$

and

$$[A] = \left([I] - (qS/\bar{c}) \left[C_{hs}\right] [a]\right)^{-1} \tag{10}$$

From Eq. (8) not only may the structural loads be calculated (i.e., shear, moment, and torque), but also the aeroelastic stability derivatives may be evaluated (e.g., the lift curve slope is given by $C_{Z_\alpha} = [I] \left\{F_a\right\}/qS\alpha$ when we take $\left\{h_r\right\} = \alpha \left\{x\right\}$ in the case of a symmetrical angle of attack).

We next consider the flutter problem. A flutter analysis is usually carried out by assuming a harmonic solution and seeking the combination of speed and frequency for which harmonic motion actually exists. To this end, an artificial structural damping is introduced into the system to sustain the assumed harmonic motion, and the flutter point is found when the actual damping and the required damping are equal. The total applied loads require the addition of the inertial loads to the oscillatory aerodynamic loads of Eq. (2), which for the assumed harmonic solution leads to

$$\left\{F_a\right\} = -[M] \left\{\ddot{h}_f\right\} + \left\{F\right\} \tag{11a}$$

$$= \omega^2 \left([M] + \rho b_r^2 s \left[C_h\right]\right) \left\{h_f\right\} \tag{11b}$$

Combining Eqs. (6) and (11b), and introducing the artificial damping as a complex divisor (a complex factor to the modulus of elasticity) to the flexibility matrix, leads to the formulation of the flutter eigenvalue problem

$$\{h_f\} = \frac{\omega^2}{1 + ig}[a]\left([M] + \rho b_r^2 s [C_h]\right)\{h_f\} \quad . \tag{12}$$

The eigenvalues, $\lambda = \lambda_R + i\lambda_I = (1 + ig)/\omega^2$, of Eq. (12) may be found by complex matrix iteration from which we find the required damping

$$g = \lambda_I/\lambda_R \quad , \tag{13}$$

the frequency

$$\omega = 1/\sqrt{\lambda_R} \quad . \tag{14}$$

and, since the formulation of the oscillatory AICs required the assumption of a reduced frequency, the velocity

$$V = \omega b_r/k_r \quad . \tag{15}$$

The frequency response problem requires the addition of the harmonic gust forces of Eq. (3) to the oscillatory applied loads of Eq. (11b). The total applied force becomes

$$\{F_a\} = \omega^2\left([M] + \rho b_r^2 s[C_h]\right)\{h_f\} + \{F_g\} \quad . \tag{16}$$

Substituting Eq. (16) into Eq. (6), and again introducing the structural damping, but here as its actual value, permits the solution for the frequency response of the structural deformation mode

$$\{h_f\} = \rho V W_g b_r s[R]\left[C_g\right]\{w_g/W_g\} \tag{17}$$

where

$$[R] = \left([I] - (\omega^2/(1 + ig_a))[a]\,([M] + \rho b_r^2 s[C_h])\right)^{-1} \quad . \tag{18}$$

If the response calculation is carried out for constant velocity, then the range of reduced frequency used to determine the oscillatory AICs for motion and for the

gust follows from the range of frequency $(\omega_1 \leq \omega \leq \omega_N)$ of interest

$$\omega_1 b_r / V \leq k_r \leq \omega_N b_r / V \tag{19}$$

From the frequency response of the structural deformations, the frequency response of any stresses may be found, and a basis is provided for random loading fatigue calculations.

The above examples have shown the application of AICs in a collocation formulation of various aeroelastic problems. But a collocation solution is not always feasible when a very large number of degrees of freedom must be treated and a series or modal solution of the collocation equations must be sought. The modal formulation may be derived a number of ways (e.g., from the collocation equations using the Galerkin method, or from the mass, stiffness, and AIC matrices using the Lagrange method). We do not wish to illustrate the modal formulations of the preceding examples, for these are perhaps better known than the collocation formulations. However, we do wish to illustrate the calculation of the generalized aerodynamic forces from the AICs for use in the modal approach. Consider a series approximation to the deflection mode,

$$\left\{h\right\} = \left[h_n\right] \left\{q_n\right\} \tag{20}$$

where $\left[h_n\right]$ is a rectangular matrix of the assumed modes (e.g., each column may be one of the free-vibration modes), and the q_n are their respective generalized coordinates. We may compute the generalized force in each mode by considering the virtual work done in perturbing the mode. The virtual work done in perturbing the m'th mode is given by

$$\delta W_m = \left\{\delta h_m\right\}^* \left\{F\right\} \quad . \tag{21}$$

and the generalized force is given by

$$Q_m = \delta W_m / \delta q_m = \left\{\delta h_m / \delta q_m\right\}^* \left\{F\right\} \quad . \tag{22}$$

We note from the assumed series, Eq. (20), that

$$\left\{ \delta h_m / \delta q_m \right\} = \left\{ h_m \right\} \tag{23}$$

with which we obtain

$$Q_m = \left\{ h_m \right\}^* \left\{ F \right\} \quad . \tag{24}$$

We may extend this result to obtain all of the generalized forces

$$\left\{ Q \right\} = \left[h_n \right]^* \left\{ F \right\} \quad . \tag{25}$$

Combining Eqs. (20) and (25) first with Eq. (1) leads to the static generalized forces

$$\left\{ Q_s \right\} = (qS/\bar{c}) \left[h_n \right]^* \left[C_{hs} \right] \left[h_n \right] \left\{ q_n \right\} \quad , \tag{26}$$

and then with Eq. (2) leads to the oscillatory generalized forces for motion

$$\left\{ Q \right\} = \rho \omega^2 b_r^2 s \left[h_n \right]^* \left[C_h \right] \left[h_n \right] \left\{ q_n \right\} \quad ; \tag{27}$$

combining Eqs. (3) and (25) leads to the oscillatory generalized gust forces

$$\left\{ Q_g \right\} = \rho V W_g b_r s \left[h_n \right]^* \left[C_g \right] \left\{ w_g / W_g \right\} \quad . \tag{28}$$

At this point, it is obvious that the AICs offer a *large* computational advantage in aeroelastic analyses by modal methods, since the AICs need be derived *only once*, and the generalized forces follow from multiplications with the modal matrix; any appraoch that derives generalized forces directly is *inefficient* to the extent that the aerodynamic analysis must be repeated for *every* revision of the modal set.

The discussion throughout this paper will center on theoretical results. We cannot discuss experimental correlation of the theoretical AICs, since sufficient

unsteady measurements are not available. Any modification of theoretical unsteady AICs must be based on the limiting case of the static AICs and corresponding steady wind tunnel measurements. A derivation of a weighting matrix that permits adjustment of the theoretical AICs to incorporate the results of experimental measurements is presented in Section V.

II. AICs FOR MOTION

A. Strip Theory

We consider first the derivation of the AICs from the simplest of the aerodynamic theories - - that of strip theory in which the flow along any section of the wing can be considered as two-dimensional. The derivation is taken from Ref. 1, and is repeated here to illustrate the three basic relationships that must be established for any aerodynamic theory to obtain the AICs: (1) the pressure-downwash relation; (2) the force-pressure relation; and (3) the downwash-deflection relation. Although these are the basic relationships, they are found in an equivalent manner in the case of strip theory, since the pressure integration has already been carried out for the tabulation of the oscillatory coefficients. However, this equivalence is apparent, and, since the strip theory approach is so fundamental, we shall find it a satisfactory introductory illustration.

In the case of strip theory, the matrix of AICs appears in a partitioned form. For example, the AICs for the two-strip wing appear as

$$\left[C_h\right] = \left[\begin{array}{c|c} C_{h1} & 0 \\ \hline 0 & C_{h2} \end{array}\right] \tag{29}$$

where the C_{hi} are the AICs for each strip. Hence, it is only necessary to derive the AICs for one strip. The given and the equivalent force systems and geometry are shown in Fig. 1. We have arbitrarily chosen the forward control point at the quarter-chord, the intermediate control point at the control-surface hinge line, and the aft control point at the trailing edge. The chord is considered to be rigid. The oscillatory aerodynamic coefficients referred to the quarter-chord are defined by

$$\begin{Bmatrix} L \\ M \\ T \end{Bmatrix} = \pi \cos \Lambda \, \rho\omega^2 b^2 \Delta y \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix} \begin{bmatrix} L_h & L_\alpha & L_\beta \\ M_h & M_\alpha & M_\beta \\ T_h & T_\alpha & T_\beta \end{bmatrix} \begin{Bmatrix} h \\ b\alpha \\ b\beta \end{Bmatrix} \tag{30}$$

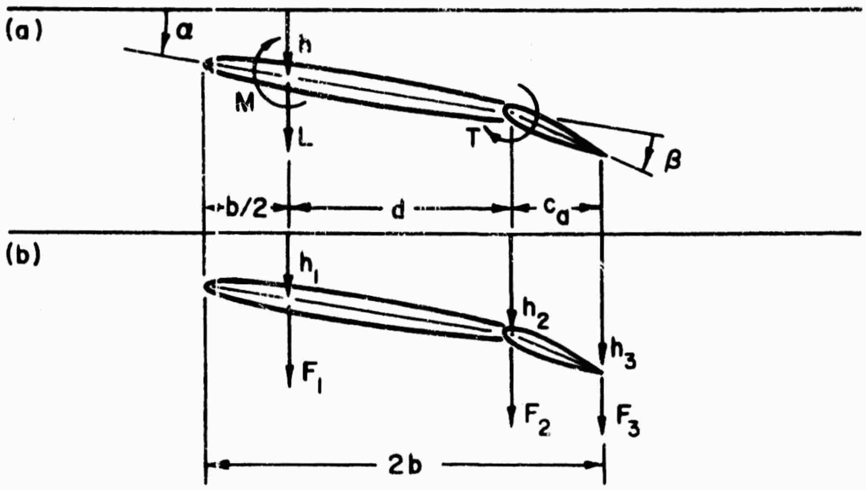

Fig. 1. The Original (a) and the Equivalent (b) Force Systems and Geometry for Strip Theory

(N.B., the factor $\pi \cos \Lambda$ is used with subsonic coefficients; usually the factor 4 is used in its place with supersonic coefficients.) Eq. (30) is the strip theory equivalent of the basic relationship between pressure and downwash. The equivalent of the force-pressure relationship arises from the transformation of the given loads to the control point forces. This load equivalence is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & d & (d + c_a) \\ 0 & 0 & c_a \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} L \\ M \\ T \end{Bmatrix} . \tag{31}$$

The geometrical equivalence, which is the analog of the downwash-deflection relationship, is given by

$$\begin{Bmatrix} h \\ b\alpha \\ b\beta \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -b/d & b/d & 0 \\ b/d & -(b/d + b/c_a) & b/c_a \end{bmatrix} . \tag{32}$$

Placing $\lfloor F \rfloor$ in terms of $\lfloor h \rfloor$ by combining Eqs. (30) to (32) and identifying the result with Eq. (2) yields the strip AIC matrix

$$\left[C_h\right] = \pi \cos\Lambda (b/b_r)^2 \; (\Delta y/s)$$

$$\times \begin{bmatrix} 1 & -b/d & b/d \\ 0 & b/d & -(b/d + b/c_a) \\ 0 & 0 & b/c_a \end{bmatrix} \begin{bmatrix} L_h & L_\alpha & L_\beta \\ M_h & M_\alpha & M_\beta \\ T_h & T_\alpha & T_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -b/d & b/d & 0 \\ b/d & -(b/d + b/c_a) & b/c_a \end{bmatrix} \tag{33}$$

In the absence of the control surface, we have

$$C_h = \pi \cos\Lambda (b/b_r)^2 \; (\Delta y/s) \begin{bmatrix} 1 & -b/d \\ 0 & b/d \end{bmatrix} \begin{bmatrix} L_h & L_\alpha \\ M_h & M_\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -b/d & b/d \end{bmatrix} . \tag{34}$$

Once the AIC partitions are obtained for each strip, the surface AICs are formed by assembling the partitions according to Eq. (29).

A survey of the status of two-dimensional oscillatory aerodynamic theory must begin with the incompressible solutions of Theodorsen[13] and of Theodorsen and Garrick[14, 15] and the tabulations of Smilg and Wasserman.[16] These solutions are for the case of a rigid airfoil and control surface; the incompressible solution for the case of a flexible chord undergoing parabolic changes in camber has been obtained by Spielberg.[17] The subsonic solution for the rigid airfoil and control surface has been obtained and tabulated by Timman,

van de Vooren, and Griedanus.[18] ... sonic solution has been given by Rott,[19] and the extension to include a ... trol surface and a tabulation have been given by Nelson and Berman.[20] ... supersonic solution for a flat plate airfoil including a control sur... has been obtained by Garrick and Rubinow[21] and tabulated by Hu... nd Durling.[22] The zero thickness supersonic solution tends to gi... nservative flutter predictions because of its aft location of ae... ic center. A supersonic frequency expansion solution for the air... pressure including thickness effects but without a control surface has ... given by VanDyke,[23] and the pressure coefficient has been integrated by R... den and Revell[24] to obtain the oscillatory coefficients; the results have not b... en tabulated. The solution with thickness effects approaches second-order piston theory at high Mach number. Third-order piston theory including a control surface has been discussed by Ashley and Zartarian;[25] this solution will be reviewed in a three-dimensional context in a later section.

B. Lifting Surface Theory

1. Subsonic Lifting Surface Theory

a. Introduction

In the following sections we deduce the matrix equations for the aerodynamic influence coeffic... lifting surface theo...s. Representing the control point force to deflection relation to be as previously defined in Section I,

$$\left\{F_j\right\} = \rho\omega^2 b_r^2 \ell \left[C_h^j\right] \left\{h_j\right\} \tag{35}$$

then we shall show that a convenient representation is

$$\left[C_h^j\right] = K\,(1/k_r^2)\left[B_{nm}^j\right]\left[A_{nm}^i\right]\left[W_{ij}\right] \tag{36}$$

where $\left[B_{nm}^j\right]$ is a pressure integration matrix, $\left[A_{nm}^i\right]$ is the matrix relating pressure loading coefficients* to downwash, and $\left[W_{ij}\right]$ relates downwash to deflections of the collocation points. We first consider these component matrices in detail.

*We refer to methods leading to an integral equation relating downwash and the pressure loading on the wing represented by a series of functions with unknown coefficients to be determined as the main problem.

b. Review of Earlier Solutions

The general theoretical foundation for the solution of the subsonic unsteady problem was laid by work of Kuessner,[26] and, subsequently, by Watkins, Runyan, and Woolston.[27] With the advent of large high-speed digital computers, the development of numerical methods followed. Among these was the method of Runyan and Woolston,[28] which was subsequently developed by Rodden, Hodson, and Revell[29] in the form of AICs according to the present formulation. This development will be discussed in the context of the general method to illustrate the technique. The more recent work of Watkins, Woolston, and Cunningham,[30] having more general applicability, forms the basis for recent developments of Revell, Carrington, Milligan, and Vivian.[31] A development basically parallel to the NASA work has been made by Hsu[32, 33] and the M. I. T. Aeroelastic Research Laboratory (see Ref. 34 for a general review of this work). All of these subsonic methods are based on the acceleration potential, or pressure loading approach, wherein the unknown pressure distribution on the planform is represented by a series of functions whose unknown coefficients are related to the known downwash distribution by an integral equation, which for the harmonic case is of the form

$$w(x, y, 0, \omega) = \iint_S K(x - \xi, y - \eta, M, \omega)\, \Delta p(\xi, \eta)\, d\xi\, d\eta \tag{37}$$

and

$$\Delta p(\xi, \eta) = \sum_{m=0}^{M} \sum_{n=0}^{N} \Delta p_{nm}(\xi, \eta)\, a_{nm} \quad . \tag{38}$$

We consider the development of AICs as illustrated by the earlier development of Rodden, Hodson, and Revell[29] in the context of the current work of Ref. 31.

The derivation of the AICs from an arbitrary lifting surface theory is always possible, however complex may be the details. We present a systematic approach which is valid for either subsonic or supersonic flow when the acceleration potential method is used. This form is preferred over the velocity potential method in supersonic cases involving subsonic edges as will be discussed in Section II. B. 3). We define the downwashes $\{w_i/V\}$, $(i = 1(1)N_i)$, induced at

a point set $\{i\}$ in the field to be linearly related to the load distribution expressed as a series of suitable functions and undetermined coefficients $\{a_{nm}\}$, $(n = 1(1)N,\ \ m = 1(1)M)$. In matrix form, we write

$$\{w_i/V\} = [D^i_{nm}]\ \{a_{nm}\} \tag{39}$$

where in the subsonic case, following Watkins, Woolston, and Cunningham

$$D^i_{nm} = \frac{b_o \ell}{4\pi} \int_{-1}^{1} d\eta \int_{\xi_{\ell.c.}(\eta)}^{\xi_{t.e.}(\eta)} \frac{K_\theta\ (x_i - \xi,\ y_i - \eta,\ M,\ k)}{\ell^2\ (y_i - \eta)^2} \frac{\Delta p_{nm}\ (\xi, \eta)\ d\xi}{\rho\ V^2} \tag{40}$$

and

$$\Delta p_{nm}/\rho V^2 = 4\pi(\ell/b_0) \left[(b_0/b) \sum_{m=0}^{M} \eta^m \sqrt{i - \eta^2} \sum_{n=0}^{N} \ell_n(\theta) \right] \tag{41}$$

The planform geometry is shown in Fig. 2, and

$$\xi = \xi_M(\eta) - (b(\eta)/b_0) \cos\theta \tag{42}$$

Before discussing the development of AICs, we pause to consider the problem posed by Eq. (35) which is the heart of the aerodynamic problem.

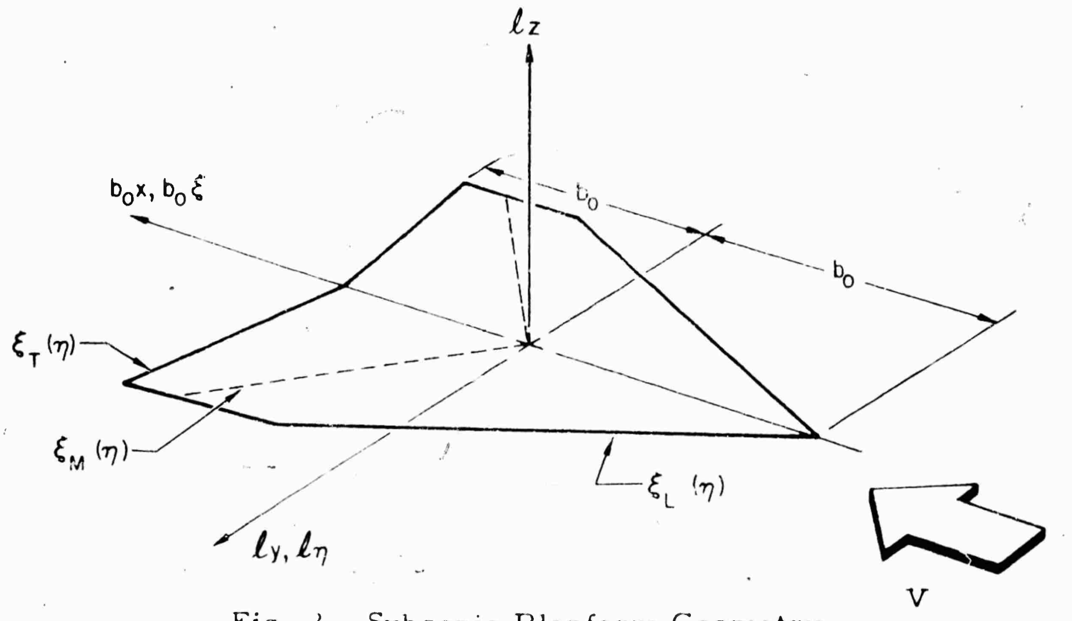

Fig. 2. Subsonic Planform Geometry

c. Calculation of the Induced Downwash

The matrix $[D^i_{nm}]$ of Eq. (39) is handled several ways. The Runyan and Woolston method follows the approach of replacing the distributed loads by a discrete set of concentrated loads $\left\{\Delta P_\nu(x_\nu, \eta_\nu)\right\}$ and related them to the downwash at the control grid $\left\{i\right\}$ by a set of "downwash factors" $[F_{\nu N}/\epsilon_N]$ for a spanwise strip ϵ_N wide, so that

$$\left\{w_i/V\right\} = (1/4\pi\rho V^2)\,[F_{\nu N}/\epsilon_N]\left\{\Delta P_\nu(x_\nu, \eta_\nu)\right\} \quad . \tag{43}$$

Now the concentrated loads are related by an integration matrix $[g_m(\eta)G_{\nu n}]$ to the pressure loading coefficients $\left\{a_{nm}\right\}$ so that

$$\left\{\Delta P_\nu(x_\nu, \eta_\nu)\right\} = 4\ell\rho V^2\left[g_m(\eta_\nu)G_{\nu n}\right]\left\{a_{nm}\right\} \tag{44}$$

where from Eq. (41) we have

$$g_m(\eta) = \eta^m\sqrt{1-\eta^2} \quad . \tag{45}$$

Then Eqs. (43), (44), and (39) imply that

$$\left[D^i_{nm}\right] = (\ell/\pi)\left[F_{\nu N}/\epsilon_N\right]\left[g_m(\eta_\nu)G_{\nu n}\right] \quad . \tag{46}$$

In the method of Runyan and Woolston, and, hence, the above developments, the downwash factors are expanded in a fifth-degree polynomial in the reduced frequency $k = \omega b_0/V$. Furthermore, the fifth-degree term behaves as k^5/β^8 as $M \to 1$; whence, the solution is only valid when $kM/\beta^2 << 1$, although it should serve as adequate for many lower mode instability problems for Mach numbers up to, say 0.80. Application of the ratio test shows convergence whenever $|kx_0| < 10\beta^2/3$, which, in general, is nonuniform on the planform (through the x_0 variation). However, for delta wings, $|x_0| < 2$, and $k < 5\beta^2/3$ is a sufficient condition for uniform, absolute convergence.

The method of handling $[G_{\nu n}]$ in Ref. 29 differs somewhat from that of Runyan and Woolston, but leads to similar quantitative results. An improved computational procedure for this method has been developed by Rodden, Farkas, and Oyama.[35]

The more recent works of Watkins, Woolston, and Cunningham[30] and Hsu[32, 33] proceed to evaluate $\left[D^{i}_{nm}\right]$ directly by Gaussian quadrature for the nonsingular regions, and uses the finite part concept of Hadamard to treat the singular region (see Ref. 36, pp. 277-280). The development of Revell, Carrington, Milligan, and Vivian[31] considers piecewise continuous planform edges, and the integration must be interpreted in the Lebesque sense. The integration scheme of Ref. 30 is essentially valid over the entire frequency range, and the more accurate integration provides a more reliable representation of the effect of the distributed loading than does the replacement loads method. This $\left[D^{i}_{nm}\right]$ matrix is the point of departure for the remainder of the problem, and the question arises as to how many loading functions and downwash match points to demand for an adequate, efficient engineering solution of the problem.

d. <u>Numerical Solution of the Downwash Integral Equation</u>

An exact solution of the linear boundary value problem of lifting surface theory demands that the downwash shall match some prescribed values everywhere on the planform. The boundary condition, therefore, requires $N_i \rightarrow \infty$. For computational purposes we settle for the finite number of points $P_i(x_i, y_i)$. Historically, the early computational procedures developed by Watkins, Woolston, and Cunningham have chosen $N_i = 9$, and 16. The method of Hsu[32, 33] involves a 25-point collocation scheme whose locations are chosen to preserve the total chordwise load. In order to solve Eq. (39), it is clear that we need one equation for each unknown a_{nm}. Thus the simplest view of the problem is to set $N_i = N \times M$ and make N_i as large as possible for a given digital computing capability. However, because of the polynomial form of $g_m(\eta)$, the matrix $\left[D^{i}_{nm}\right]$ is not "well conditioned" for a finite digit arithmetic system. A common remedy to such problems is the least square solution which overdetermines the problem (see Ref. 37, pp. 149-164). Thus, we shall choose a relatively small value of $N \times M$ which is sufficient to describe our load distribution, but endeavor to increase N_i as far as possible. We thus find a nonfinality of the $\left\{a_{nm}\right\}$, but for <u>each combination</u> N & M we find a <u>convergence</u> of the $\left\{a_{nm}\right\}$ as $N_i \rightarrow \infty$.

We thus expect to find an optimum computational solution by choosing $(N \times M) < N_i$ to reduce the size of the associated algebraic problem. This point is clear when we consider the general solution of Eq. (39) for the case $N_i > (N \times M)$. Here we have

$$\left\{ a_{nm} \right\} = \left[A^i_{nm} \right] \left\{ w_i / V \right\} \quad , \tag{47a}$$

where from Eq. (39)

$$\left[A^i_{nm} \right] = \left(\left[D^i_{nm} \right]^* \left[\Delta S_j \right] \left[D^i_{nm} \right] \right)^{-1} \left[D^i_{nm} \right]^* \left[\Delta S_j \right] \tag{47b}$$

where $\left[\Delta S_j \right]$ is the least squares diagonal weighting matrix whose elements consist of the incremental areas surrounding the control points. We note that $\left[D^i_{nm} \right]$ is complex for the oscillatory aerodynamic problem, and it can be considered as the Fourier transform of the general time dependent downwash. Noting the ranks of the matrices carefully, we know that the solution of Eq. (47) exists only if $N_i > (N \times M)$. Physically speaking we are approximating the N_i point boundary condition in a least square error sense for any given $N \times M$ combination of loading functions. We shall now discuss the loading functions briefly.

e. Choice of Loading Functions

(1) Chordwise Functions: The most widely used loading functions satisfy the Kutta condition of pressure continuity at the trailing edge, and have the classical, thin airfoil square root singularity at the leading edge. In terms of the trigonometric variable, θ, defined by Eq. (42) the $\left\{ \ell_n(\theta) \right\}$ of Eq. (41) are of the form

$$\ell_0 (\theta) = \text{ctn} (\theta/2) \tag{48a}$$

$$\ell_n (\theta) = (4/2^{2n}) \sin (n\theta), \qquad n \geq 1 \quad . \tag{48b}$$

In the case of a trailing edge flap, we have a discontinuity in downwash, and Hsu[32] has shown that a logarithmically singular loading term arises of the form

$$\ell_c(\tilde{x}) = \frac{\Delta w_a \sqrt{1-M^2}}{2\pi^2 (\ell/b_0)} \sqrt{\frac{1-\tilde{x}}{1+\tilde{x}}} \left\{ -\frac{1}{2} \log \left| \frac{1 - c\tilde{x} + \sqrt{1-c^2}\sqrt{1-\tilde{x}^2}}{1 - c\tilde{x} - \sqrt{1-c^2}\sqrt{1-\tilde{x}^2}} \right| - \left(\frac{\pi}{2} - \sin^{-1} c\right) \right\} \tag{49}$$

for a control surface covering $c \leq \tilde{x} \leq 1$, and the wing chord is defined by $-1 \leq \tilde{x} \leq 1$.

(2) Spanwise Loadings; Interference Effects: For the treatment of interference effects it is convenient to modify the spanwise loading function given by Eq. (45), for clearly at the wing root, no combination of functions of the type $\eta^m \sqrt{1-\eta^2}$ will alter the loading for $\eta \ll 1$. Falkner[4] has contrived the so-called P functions for this purpose. However, for the purposes of this discussion, any convenient analytic function which is even in $\eta - \eta_d$, where η_d is the center of a spanwise interference effect, would be sufficient. We might choose, for example

$$g_m(\eta, \eta_d) = (1/2\,M\ell)\sqrt{1-\eta^2}\,\cos m\pi(\eta - \eta_d) \tag{50}$$

where $M > m$, or

$$g_m(\eta, \eta_d) = \left[1 - (\eta - \eta_d)^{2m}\right]\sqrt{1-\eta^2} \tag{51}$$

Having described the classes of convenient subsonic loading functions which satisfy the boundary condition, we consider the problem of downwash specification.

f. Specification of the Downwash; Interference Considerations

The planform boundary condition for the general aerodynamic problem is to match the normal component of velocity of the fluid relative to the surface

$$\vec{V}_b \cdot \vec{n} = \vec{V}_f \cdot \vec{n} \quad , \tag{52}$$

and for the linearized lifting surface problem, this amounts to evaluating w_f on the plane $z = 0$, (cf., for example, Miles[38]). Hence, Eq. (39) expresses the fluid downwash velocities, w_{fi}, induced by the wing loading at any point, (x_i, y_i) in the plane $z = 0$. The body vertical velocity, at (x_i, y_i), therefore, must equal the sum of all the induced fluid downwash velocities from each of N_d disturbance fields (of which, the wing-induced downwash is the major

component). Then

$$w_{bi} = \sum_{n=1}^{N_d} (w_{fi})_n \quad . \tag{53}$$

The body downwash for an arbitrary surface deformation pattern $Z_w = h(x, y, t)$ is to the first order of approximation,

$$w_{bi} = \left(V \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} \right)_{x,\, y = x_i,\, y_i} \quad . \tag{54}$$

For the purpose of the collocation method as discussed in the previous sections, we must relate $\left\{ w_i / V \right\}$ to the deflections at a discrete grid point set $\left\{ h_j \right\}$ ($j = 1\ (1) N_j$). To this end we define $\left[W_{ij} \right]$ such that

$$\left\{ w_{bi}/V \right\} = (1/b_0) \left[W_{ij} \right] \left\{ h_j \right\} \quad . \tag{55}$$

The problem of computing $\left[W_{ij} \right]$ requires some type of analytic interpolation of the deflections. For example, let

$$h\,(x, y) = \sum_r \sum_s d_{rs} H_{rs}\,(x, y) \quad . \tag{56}$$

At our control point grid we have

$$\left\{ h_j \right\} = \left[H^j_{rs} \right] \left\{ d_{rs} \right\} \quad . \tag{57}$$

Using a local chordwise interpolation only, we obtain for $y = y_1$, $x = x_j$

$$\left\{ h_j\,(y_1) \right\} = \left[H^j_r\,(x_j) \right] \left\{ d_r\,(y_1) \right\}, \quad j = 1\ (1)\ N_j\,(y_1) \quad . \tag{58}$$

Then by least squares

$$\left\{ d_r\,(y_1) \right\} = \left(\left[H^j_r \right]^* \left[\Delta x_j \right] \left[H^j_r \right] \right)^{-1} \left[H^j_r \right] \left[\Delta x_j \right] \left\{ h_j \right\} \tag{59a}$$

$$= \left[I^j_r \right] \left\{ h_j \right\} \tag{59b}$$

where the least squares diagonal weighting matrix is composed of the incremental chord lengths Δx_j. Then Eqs. (59) and (55) imply

$$\left[W_{ij}(y_1)\right] = \left[\frac{\partial H_r}{\partial(x/b_0)} + ikH_r\right]_{x=x_i} \left[I_r^j\right] \tag{60}$$

A general discussion of this problem would lead us too far afield, but it is an important problem. We merely remark here that a local chord or interpolation "in the small" is most advisable with considerable "overdetermination" of the $\left\{d_r\right\}$ being a necessary requisite for reliable results. Interpolation of the entire surface ("in the large") can often be done but frequently leads to nearly singular matrices.

To consider an interference problem, we specialize Eq. (53) to the case of a wing and a fuselage and write using Eq. (39),

$$\left\{w_{bi}^{W}/V\right\} = \left\{w_i^{W}/V\right\} + \left\{w_i^{F}/V\right\} \tag{61a}$$

$$= \left[D_{nm}^{iW}\right]\left\{a_{nm}^{W}\right\} + \left\{w_i^{F}/V\right\} \quad . \tag{61b}$$

Then our solution for Eq. (47) requires that we define $\left\{w_i/V\right\}$ as

$$\left\{w_i^{W}/V\right\} = \left\{w_{bi}^{W}/V\right\} - \left\{w_i^{WF}/V\right\} \tag{62a}$$

$$= (1/b_0)\left[W_{ij}^{W}\right]\left\{h_j^{W}\right\} - \left\{w_i^{WF}/V\right\} \tag{62b}$$

where $\left\{w_i^{WF}/V\right\}$ might be evaluated from some separate analysis and related to the local fuselage downwash and deflections. For example, by slender body theory the fuselage field perturbation potential is (cf., Miles[38]), noting $\sin\theta = z/r$ for a circular fuselage

$$\phi^F(x,t;\, r,\theta) = -(1/r)\, w_b^F(x,t)\, R_F^2(x)\sin\theta \quad , \tag{63}$$

whence, the downwash in the field is

$$w_i^{WF}(x_i;\, r_i, \theta_i, t) = \partial\phi^F/\partial z \tag{64a}$$

$$= (1/r_i^2)\, w_b^F (x_i, t)\, R_F^2 (x_i)\, (1 - 2 z_i^2/r_i^2) \quad . \tag{64b}$$

For $z_i^2 \ll r_i^2$

$$w_i^{WF} = w_b^F (x_i, t)\, R_F^2 (x_i)/r_i^2 \quad . \tag{64c}$$

Further if the fuselage downwash is related to the deflections by

$$\left\{ w_i^{WF} (x_i, t) \right\} = (1/b_0) \left[W_{ij}^F \right] \left\{ h_j^F \right\} \tag{65}$$

we have again for $z_i^2 \ll r_i^2$,

$$\left\{ w_i^{WF} (x_i, t;\, y_i, 0)/V \right\} = (1/b_0) \left[R_F^2 (x_i)/r_i^2 \right] \left[W_{ij}^F \right] \left\{ h_j^F \right\} \tag{66}$$

Then substituting Eq. (66) into Eq. (62b) gives an aerodynamic interference coupling between the fuselage and wing motion

$$\left\{ w_i^W / V \right\} = (1/b_0) \left(\left[W_{ij}^W \right] \left\{ h_j^W \right\} - \left[R_F^2 (x_i)/r_i^2 \right] \left[W_{ij}^F \right] \left\{ h_j^F \right\} \right) \tag{67a}$$

which may be written in the partitioned form

$$\left\{ w_i^W / V \right\} = \left[W_{ij}^{WW} \;\vdots\; W_{ij}^{WF} \right] \begin{Bmatrix} h_j^W \\ \hline h_j^F \end{Bmatrix} \quad . \tag{67b}$$

The matrices $\left[W_{ij}^W \right]$ and $\left[W_{ij}^{WF} \right]$ are obtained by means of deflection interpolation as discussed previously. We next consider development of the matrix $\left[B_{nm}^j \right]$ relating the forces $\left\{ F_j \right\}$ to the loading coefficients, $\left\{ a_{nm} \right\}$.

g. Relation Between Control Point Forces and Pressure Loading Coefficients

This is perhaps the most intuitively simple aspect of the method, and in the limit of an infinite number of force points gives $\left[\text{cf., Eq. (41)} \right]$

$$F_j \sim \Delta p_j\, \Delta S_j \sim \rho V^2\, \ell_n (\theta_j)\, g_m (\eta_j)\, \Delta S_j \quad . \tag{68}$$

However, what we are interested in is a finite lumping of the aerodynamic loads which, in some sense, preserves the most important properties of the distributed load and must be placed at points chosen arbitrarily by the dynamics engineer (for reasons of inertia and stiffness data availability). We formalize the definition of $\left[B_{nm}^j \right]$ by the relation

$$\left\{F_j\right\} = \pi\rho V^2 \ell^2 \left[B^j_{nm}\right] \left\{a_{nm}\right\} \quad . \tag{69}$$

The simplest treatment, and one which is especially appropriate for a large number of chordwise collocation stations, is to write

$$\left[B^j_{nm}\right] = (1/\pi\rho V^2 \ell^2) \iint_{\Delta S_j} \Delta p_{nm}\, b_0\, \ell\, d\xi\, d\eta, \; j = 1\,(1)\, N_j \quad . \tag{70}$$

In the method of Ref. 31 for the cases where $4 \leq N_j(y) \leq 5$, the strip lift, pitching moment, and control surface hinge moment are matched, and, rather than matching higher order moments, an alternative method called the "minimum vector length constraint" is used. Although the method of matching higher order moments is necessary to yield the correct generalized forces for an arbitrary chordwise deformation mode, it does not lead to a satisfactory estimation of the chordwise load distribution for the purpose of stress analysis when only a small $\left[\text{e.g., } N_j(y) = 5\right]$ number of chordwise control points are chosen. The minimum vector length constraint adds to the basic conditions of lift, pitching moment, and hinge moment, the requirement that the sum of the squares of the control point forces be a minimum. We estimate that the procedure defined by Eq. (70) is satisfactory for $N_j(y)$ of the order of six or more per half wave of chordwise deformation, and the minimum vector length constraint approach has its utility in nearly rigid chordwise deformation problems. We add that one must keep a balanced view in regard to aerodynamic lumping versus the inertial lumping for the vibration analysis, and with due consideration for the quality of the structural influence coefficients for the aeroelastic analysis.

h. The Aerodynamic Influence Coefficients

We are now in a position to assemble the component matrices to obtain the desired AICs. The fundamental relationships as described in Section II A are formed in: Eq. (47a), the pressure-downwash relation, Eq. (69), the force-pressure relation, and Eq. (55), the downwash-deflection relation. Combining these so as to relate the forces and deflections leads to

$$\left\{F_j\right\} = \pi \rho V^2 (\ell^2/b_0) \left[B^j_{nm}\right] \left[A^i_{nm}\right] \left[W_{ij}\right] \left\{h_j\right\} \quad . \tag{71}$$

Identifying Eqs. (35) and (72) leads to the AICs

$$\left[C_h\right] = \pi (1/k_r^2) (\ell /b_0) \left[B^j_{nm}\right] \left[A^i_{nm}\right] \left[W_{ij}\right] \quad . \tag{72}$$

We have elaborated on the formulation for the subsonic case because it contains most of the possible complexities of applying lifting surface theory, and we have demonstrated the feasibility of the collocation format of the AICs. We next consider lifting surface theory in other Mach number regimes.

2. Transonic Lifting Surface Theory

a. Introduction

This regime is the most important of all the Mach number ranges in aeroelasticity, even for vehicles whose design Mach numbers are in the supersonic range. In fact even for a hypersonic vehicle, there will exist two critical regimes controlling stiffness:

(1) A transonic design point, due to the fact that aerodynamic derivatives reach their maximum in this regime.

(2) A hypersonic design point, due to the forward shift of aerodynamic center due to thickness effects, and due to temperature effects on stiffness.

Perhaps the greatest source of difficulty in the transonic regime on a hypersonic vehicle is the type of configuration (e. g. , blunt leading edges and very low aspect ratios, such that the planar surface concept[38] breaks down, and we must proceed with a "wing-body" approach from the outset). However, for all vehicles for which aerodynamic heating is not crucial in dictating wing design (i. e. , approximately up to M = 3. 0), we shall face the problem of low aspect ratio, very thin wings, since wave drag is an important consideration in design. For such vehicles the transonic problem is essentially three dimensional[39] and lifting surface theory is the only reasonable basis for aeroelastic loads analysis.

b. Linearization; Shock Wave Effects

It is generally appreciated that shock waves and nonlinearity are the essential features of the steady transonic flow past a body or surface of finite thickness,[40] but it has been shown[38,39] that there exists a valid range for linearization for the unsteady regime, for low aspect ratio wings defined by[41] (see Fig. 3)

$$k \gg \left| \sigma\epsilon \quad \log\left(\sigma\epsilon^{1/3}\right) \right| \tag{73}$$

where $\sigma = 1/2b_0$, ϵ = thickness ratio, and $k = \omega b_0/V$ is the reduced frequency. (For example, for a 60-degree swept delta wing with $\epsilon = 0.03$, $\sigma = 1/\sqrt{3}$, we require $k \gg 0.0297$). In this event, the appropriate linearized equation for the unsteady motion is[38,39]

$$\phi_{1_{yy}} + \phi_{1_{zz}} - 2M^2 \phi_{1_{xt}} - M^2 \phi_{1_{tt}} = 0 \quad . \tag{74}$$

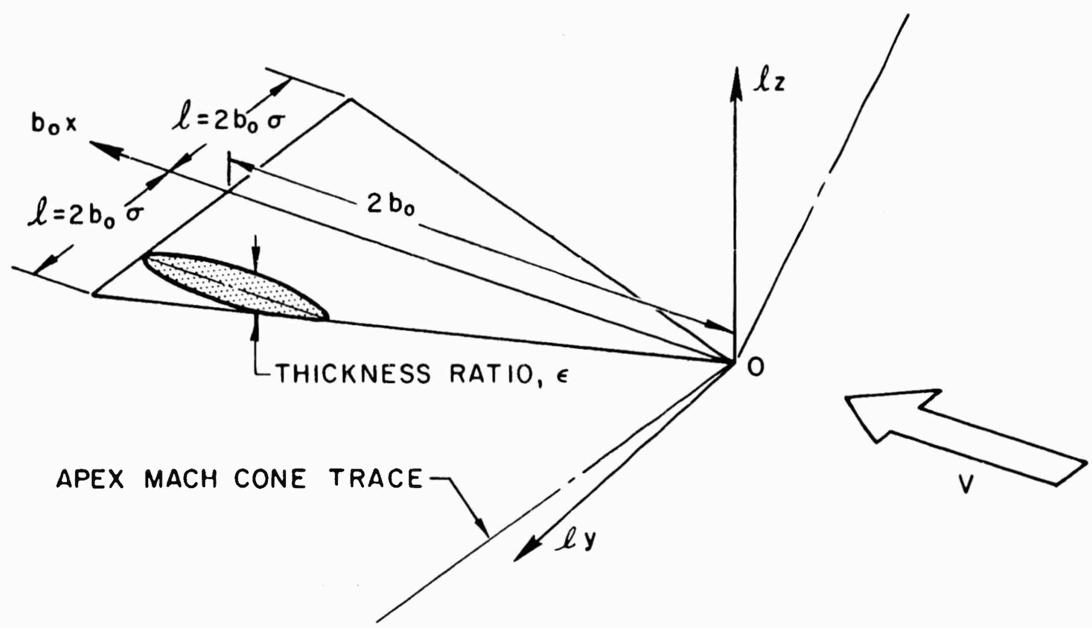

Fig. 3. Planform Geometry for Transonic Regime

The presence of shock waves has an important effect in modifying the boundary conditions. Eckhaus[42] has recently developed a solution of the idealized problem of an unsteady two-dimensional flow with a uniform supersonic zone separated from a uniform subsonic zone by a normal shock wave perpendicular to the airfoil surface (see Fig. 4a). This actually is a fairly realistic model of the subsonic-transonic regime (see Fig. 4b).

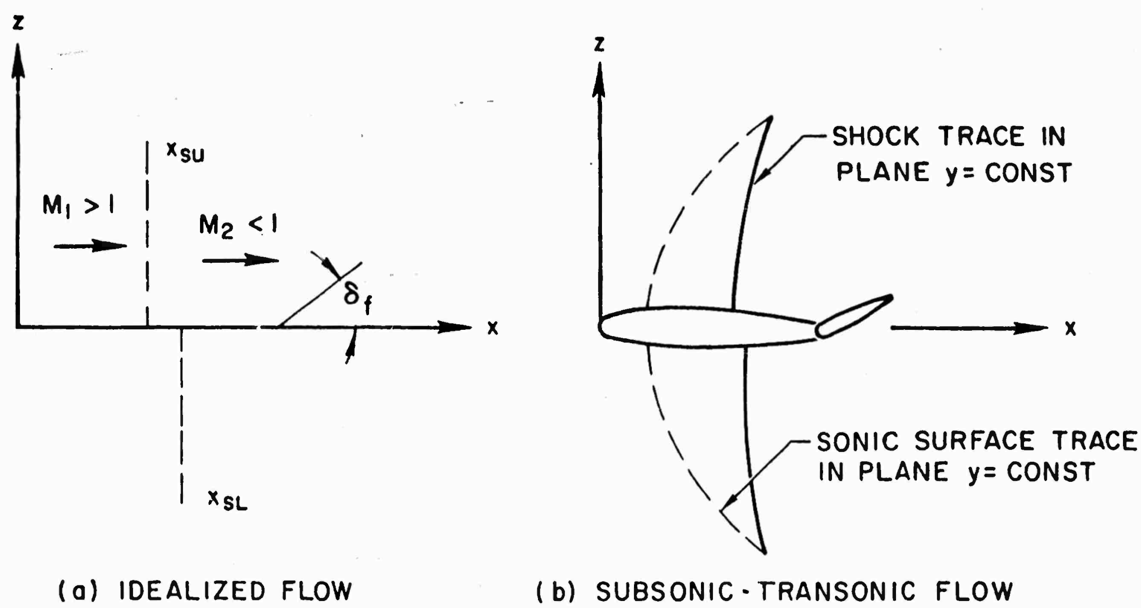

Fig. 4. Transonic Flow with Shock Waves

Eckhaus' model has already proven to be helpful in explaining many of the features of control surface buzz[39,42] wherein a disturbance due to control surface oscillation sends upstream a wave which interacts with the shock in a manner such that, as always, the Rankine-Hugoniot relations must be satisfied. For the unsteady problem these provide a set of boundary conditions relating the unsteady perturbation potential derivatives on either side of the shock.[39,42,43] Eckhaus shows that the most important first-order effect of the shock is to make the airfoil appear "semi-infinite" (with respect to an origin fixed in the steady state shock wave position). Part of the solution yields the amplitude of the shock wave oscillation in relation to the surface motion which originated the

disturbance. This is also of interest in connection with problems of shock wave interaction with turbulence which are of importance in the growing field of aerodynamic noise theory pioneered by Lighthill.[44] For a review of these turbulence interaction problems the reader is referred to Chu and Kovasznay.[45]

The Eckhaus method could provide a useful basis for the improvement of lifting surface theory as a prelude to development of AICs if:

(1) The steady state flow field about the wing body is given or can be estimated (say, for a symmetrical body);

(2) The shape and location of the shock surface is given by an equation of the form $F_s(x, y, z) = 0$;

(3) The Eckhaus solution is extended to treat the case of upstream disturbances and their transmission through the shock wave;

(4) The solution is formulated in terms of acceleration potential to avoid integration over the wake, in order to be more compatible with a unified kernel function approach.

c. Vorticity Effects

We note finally that linear methods do not preclude consideration of vorticity effects, for we may write, following Sears,[43]

$$\vec{V}_f = i\,V + \nabla\phi + \vec{V}_1 \tag{75}$$

where $\nabla \times \vec{v}_1 \neq 0$, but where

$$\left|\bar{v}_1\right| / V << 1 \; ; \; \left|\nabla\phi / V\right| << 1 \quad , \tag{76}$$

leading to equations of the form (specialized to steady flow here for illustration of the influence of steady flow nonuniformity)

$$\left[1 - M^2(y, z)\right] p'_{xx} - \frac{2M_y(y, z)}{M(y, z)} p'_y - \frac{2M_z(y, z)}{M(y, z)} p'_z + p'_{yy} + p'_{zz} = 0 \tag{77}$$

where p' is the disturbance pressure. Lighthill [46] has also introduced a modified potential function χ such that (u', v', w') is the disturbance velocity field)

$$\frac{\partial \chi}{\partial x} = p' / \rho_0 V \tag{78a}$$

$$\frac{\partial \chi}{\partial y} = v' \tag{78b}$$

and shows that for the two-dimensional case,

$$(1 - M^2)\, \chi_{xx} + \chi_{yy} = 0 \tag{79}$$

This type of consideration has been used by Miles [47] in studying boundary layer effects on panel flutter and by Lighthill[46] in shock diffraction effects (e.g., an atomic blast passing a corner of an airfoil).

The above remarks on vorticity have been included here as appropriate to the context of departures from linear potential theory, but actually, the vorticity problem <u>per se</u> can be safely ignored* for transonic aeroelastic problems (except for the proper handling of the shock boundary conditions). Thus on either side of the shock the velocity may be described by

$$\vec{V}_f = i\, U_1 + \nabla \phi_1 (x, y, z, t)\,, \qquad x < x_s \tag{80a}$$

$$\vec{V}_f = i\, U_2 + \nabla \phi_2 (x, y, z, t)\,, \qquad x > x_s \quad , \tag{80b}$$

and the isentropic pressure equation may be used,

$$\Delta p_1 = \rho_1 V_1 \phi_{1_x} + \rho_1 \phi_{1_t} \tag{81a}$$

$$\Delta p_2 = \rho_2 V_2 \phi_{2_x} + \rho_2 \phi_{2_t} \quad . \tag{81b}$$

d. Linearized Transonic Lifting Surface Theory

If we consider only the first-order effects of the shock boundary conditions, the downstream disturbances are excluded and the linearized

*The entropy is proportional to $\left\{(\gamma-1)\beta^4/M^4\left[\gamma+1-\beta^2(\gamma-1)\right]\right\}\partial p/\partial y$ and the ratio of reflected to incoming wave pressure amplitudes is $D_2/D_1 = (1-M)^2\left[3-(1+M)\right]/(1+M)^2\left[3-(1-M)\right]$ (See Ref. 42, pp. 23-25).

transonic problem is obtained. This has been thoroughly treated by Landahl.[39] He shows that a similarity property exists for solutions of Eq. (74) described by the relation

$$\phi_1 (x, y, z, t; \sigma, k, M) = (1/M)\, \phi_1 (x, M_y, M_z, t; M\sigma, k, 1) \quad . \tag{82}$$

Then the solution need be carried out only for M = 1. Ashley[48] has recently proposed a "transonic box method," using Landahl's result

$$\phi_1 (x, y, 0^+, k) = -\frac{1}{2\pi} \int_0^x d\xi \int_{-\infty}^{\infty} \frac{w(\xi, \eta)}{(x-\xi)} \exp \left\{ -\frac{ik}{2b} \left[(x-\xi) + \frac{(y-\eta)^2}{(x-\xi)} \right] \right\} d\eta \quad . \tag{83}$$

Ashley suggests that a fixed, square box grid be employed, extending out some finite distance past the wing tip, and that the diaphragm downwashes be determined algebraically in the same manner as in the supersonic application for the subsonic leading edge case. (See Section II. B. 3 for a discussion of this problem.) In this case, the fixed grid would make possible an AIC development without the inconvenience of the "floating" Mach box grid system referred to in Section II. B. 3. The above approach would constitute a practical method by which Eq. (83) may be used for general configurations.

A second approach to the linear transonic lifting surface problem is to regard it as a limiting case of the subsonic problem. Watkins, Runyan, and Woolston[27] have given the kernel for M = 1, and calculations have been carried out by the kernel function procedure. Landahl gives some comparisons with his work.[39]

We finally consider some limiting cases. For low aspect ratio and reduced frequency, we have the slender-wing theory, for which the potential satisfies Laplace's equation in planes transverse to the body axis. This solution has been developed in the form of AICs by Rodden and Revell,[49] and will be discussed at length in the following section. The other limiting case is the high-frequency slender-wing theory developed by Merbt and Landahl,[50] in which the two-dimensional wave equation in the cross-flow plane is solved in terms of Mathieu functions, after transformation to elliptic cylinder coordinates. (See Ref. 38, Ch. 9 for a review of this problem.)

Having framed the linear, transonic, unsteady lifting problem in its proper context in fluid mechanics, we now consider the formulation of AICs by slender-wing theory.

e. AICs by Slender-Wing Theory

Slender-wing theory provides a solution to the slender lifiting surface problem that achieves its maximum accuracy at transonic speeds. The present demonstration is based on Ref. 49 in which we utilize the results of the derivation of the unsteady slender-wing velocity potential given by Bisplinghoff, Ashley, and Halfman (Ref. 34, pp. 400-404). We assume for the present formulation the geometry shown in Fig. 5. The surface is divided into elemental

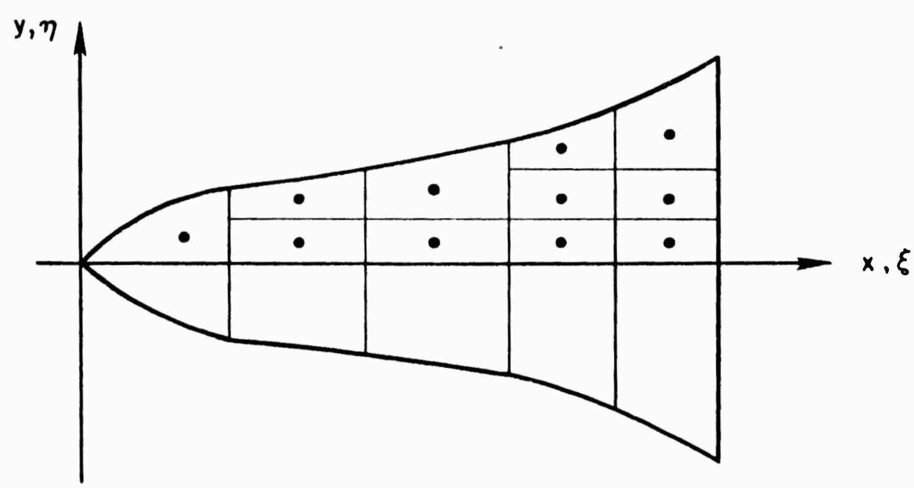

Fig. 5. Slender-Wing Geometry

areas whose deflection characteristics are specified by the deflection of their central coordinates, x_i, y_j. We nondimensionalize our coordinate system by

$$\xi,\ \eta,\ \sigma(\xi) = x/c,\ y/c,\ s(x/c)/c \tag{84}$$

and the time by

$$\tau = Vt/c \quad , \tag{85}$$

and assume that the surface deflection can be represented by

$$h(x,y,t) = c\sum\sum a_{mn}(\tau)\, f_m(\xi)\, g_n(\eta) \tag{86}$$

where

$$f_m(\xi) = \xi^m \quad \text{and} \quad g_n(\eta) = \eta^n \quad .$$

The assumed series for the deflection leads to an expression for the downwash

$$w(x, y, t) = V \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} \tag{87a}$$

$$= V \sum \sum \left[a_{mn}(\tau) f'_m(\xi) + a'_{mn} f_m(\xi) \right] g_n(\eta) \quad . \tag{87b}$$

In the present notation the perturbation of Ref. 34 becomes

$$\phi(\xi, \eta, \tau) = \frac{c}{\pi} \int_{-\sigma}^{\eta} \frac{d\nu}{\sqrt{\sigma^2 - \nu^2}} \oint_{-\sigma}^{\sigma} w(\xi, \lambda, \tau) \frac{\sqrt{\sigma^2 - \lambda^2}}{\nu - \lambda} d\lambda \tag{88a}$$

$$= c\, V \sum \sum (a_{mn} f'_m + a'_{mn} f_m) I_n(\eta, \sigma) \tag{88b}$$

where the integral I_n is given by

$$I_n(\eta, \sigma) = \frac{1}{\pi} \int_{-\sigma}^{\eta} \frac{d\nu}{\sqrt{\sigma^2 - \nu^2}} \oint_{-\sigma}^{\sigma} g_n(\lambda) \frac{\sqrt{\sigma^2 - \lambda^2}}{\nu - \lambda} d\lambda \quad . \tag{89}$$

in which the Cauchy principal part of the first integral is indicated.

The lifting pressure differential is found from the unsteady potential.

$$\Delta p = 2\rho (V\phi_x + \phi_t) \tag{90a}$$

$$= (2\rho V/c)(\phi_\xi + \phi_\tau) \tag{90b}$$

$$= 2\rho V^2 \sum \sum \left\{ \frac{\partial}{\partial \xi} \left[(a_{mn} f'_m + a'_{mn} f_m) I_n(\sigma, \eta) \right] + (a'_{mn} f'_m + a''_{mn} f_m) I_n(\sigma, \eta) \right\} \tag{90c}$$

We are now in a position to consider the force on the elemental area surrounding the point (x_i, y_j). The geometry of the elemental area is shown in Fig. 6.

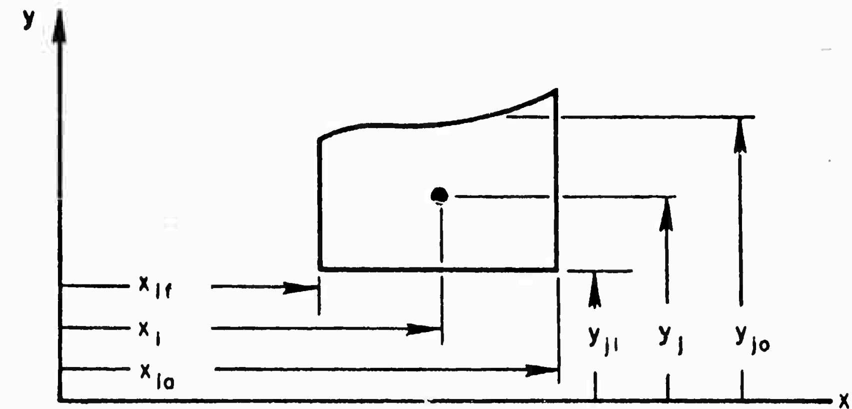

Fig. 6. Geometry of an Elemental Area

The fore and aft limits on x_i are denoted by x_{if} and x_{ia}, respectively, and the inboard and outboard limits on y_j are denoted by y_{ji} and y_{jo}, respectively. The limits x_{if} x_{ia}, and y_{ji} are considered constant. The limit y_{jo} may either be constant as in an interior region, or be variable as at the edge. The force assumed to act at the point (x_i, y_j) is found by integrating the lifting pressure over the elemental area.

$$F(x_i, y_j, t) = \iint_{\Delta S_{ij}} \Delta p(x, y, t)\, dx\, dy \tag{91a}$$

$$= c^2 \int_{\xi_{if}}^{\xi_{ia}} \int_{\eta_{ji}}^{\eta_{jo}(\xi)} \Delta p(\xi, \eta, \tau)\, d\xi\, d\eta \tag{91b}$$

where we denote by $\eta_{jo}(\xi)$ the possibility of the limit varying. The force becomes

$$F(x_i, y_j, t) = 2\rho V^2 c^2 \sum \sum \int_{\xi_{if}}^{\xi_{ia}} \int_{\eta_{ji}}^{\eta_{jo}(\xi)} \left\{ \frac{\partial}{\partial \xi} \left[(a_{mn} f'_n + a'_{mn} f_m) I_n \right] + (a'_{mn} f'_m + a''_{mn} f_m) I_n \right\} d\eta\, d\xi\ . \tag{92}$$

Since $I_n(\eta, \sigma)$ vanishes on the edge, we may interchange the order of differentiation with respect to ξ and integration with respect to η. Then, if we define

$$J_n(\xi, \eta) = \int I_n(\eta, \sigma) \, d\eta \quad , \tag{93}$$

in which we emphasize the functional dependence $\sigma = \sigma(\xi)$, and

$$J_n^j(\xi) = J_n(\xi, \eta_{jo}) - J_n(\xi, \eta_{ji}) \quad , \tag{94}$$

the force becomes

$$F(x_i, y_j, t) = 2\rho V^2 c^2 \sum\sum \int_{\xi_{if}}^{\xi_{ia}} \left\{ \frac{\partial}{\partial \xi} \left[a_{mn} f'_m + a'_{mn} f'_m) J_n^j(\xi) \right] \right.$$

$$\left. + (a'_{mn} f'_m + a''_{mn} f'_m) J_n^j(\xi) \right\} d\xi \quad . \tag{95}$$

Finally, if we define

$$K_{mn}(\xi, \eta) = \int f_m(\xi) J_n(\xi, \eta) \, d\xi \tag{96}$$

$$L_{mn}(\xi, \eta) = \int f'_m(\xi) J_n(\xi, \eta) \, d\xi \tag{97}$$

and

$$K_{mn}^j(\xi) = K_{mn}(\xi, \eta_{jo}) - K_{mn}(\xi, \eta_{ji}) \tag{98}$$

$$L_{mn}^j(\xi) = L_{mn}(\xi, \eta_{jo}) - L_{mn}(\xi, \eta_{ji}) \quad , \tag{99}$$

we obtain the final expression for the local force

$$F(x_i, y_j, t) = 2\rho V^2 c^2 \sum\sum \left\{ \left[a_{mn} f'_m(\xi_{ia}) + a'_{mn} f_m(\xi_{ia}) \right] J_n^j(\xi_{ia}) \right.$$

$$- \left[a_{mn} f'_m(\xi_{if}) + a'_{mn} f_m(\xi_{if}) \right] J_n^j(\xi_{if})$$

$$\left. + a'_{mn} \left[L_{mn}^j(\xi_{ia}) - L_{mn}^j(\xi_{if}) \right] + a''_{mn} \left[K_{mn}^j(\xi_{ia}) - K_{mn}^j(\xi_{if}) \right] \right\} \tag{100}$$

which can be abbreviated to

$$F(x_i, y_j, t) = 2\rho V^2 c^2 \sum \sum \left(M_{mn}^{ij} a_{mn} + N_{mn}^{ij} a'_{mn} + K_{mn}^{ij} a''_{mn}\right) \tag{101}$$

where

$$M_{mn}^{ij} = f'_m(\xi_{if}) J_n(\xi_{if}, \eta_{ji}) + f'_m(\xi_{ia}) J_n(\xi_{ia}, \eta_{jo}) - \left[f'_m(\xi_{if}) J_n(\xi_{if}, \eta_{jo}) + f'_m(\xi_{ia}) J_n(\xi_{ia}, \eta_{ji})\right] \tag{102}$$

$$\begin{aligned} N_{mn}^{ij} = {} & f_m(\xi_{if}) J_n(\xi_{if}, \eta_{jo}) + f_m(\xi_{ia}) J_n(\xi_{ia}, \eta_{jo}) \\ & - \left[f_m(\xi_{if}) J_n(\xi_{if}, \eta_{jo}) + f_m(\xi_{ia}) J_n(\xi_{ia}, \eta_{ji})\right] + L_{mn}(\xi_{if}, \eta_{ji}) \\ & + L_{mn}(\xi_{ia}, \eta_{jo}) - \left[L_{mn}(\xi_{if}, \eta_{jo}) + L_{mn}(\xi_{ia}, \eta_{ji})\right] \end{aligned} \tag{103}$$

$$K_{mn}^{ij} = K_{mn}(\xi_{if}, \eta_{ji}) + K_{mn}(\xi_{ia}, \eta_{jo}) - \left[K_{mn}(\xi_{if}, \eta_{jo}) + K_{mn}(\xi_{ia}, \eta_{ji})\right]. \tag{104}$$

We see from the last three relationships that the matrix elements depend on the three quantities

$$M_{mn}(\xi, \eta) = f'_m(\xi) J_n(\xi, \eta) \tag{105}$$

$$N_{mn}(\xi,\eta) = f_m(\xi) J_n(\xi,\eta) + L_{mn}(\xi,\eta) \tag{106}$$

and

$$K_{mn}(\xi,\eta) \tag{107}$$

evaluated at the four corners of each elementary area, viz., the sum of the forward inboard and aft outboard values, minus the forward outboard and aft inboard values.

Before proceeding with the derivation of the AICs let us illustrate the calculation of the various integrals by computing the lift on the rigid plunging and pitching delta wing. For pitching and plunging we have

$$h(x,y,t) = h_0(t) + x\alpha(t) \tag{108a}$$

$$= c(a_{00}f_0 g_0 + a_{10}f_1 g_0) \tag{108b}$$

or $m = 0, 1$, $n = 0$, and $a_{00} = h_0/c$, $a_{10} = \alpha$, $f_0 = 1$, $g_0 = 1$, $f_1 = \xi$.
It is first necessary to evaluate $I_0(\eta,\sigma)$.

$$I_0(\eta,\sigma) = \frac{1}{\pi}\int_{-\sigma}^{\eta} \frac{d\nu}{\sigma^2 - \nu^2} \oint_{-\sigma}^{\sigma} g_0(\lambda)\frac{\sqrt{\sigma^2-\lambda^2}}{\nu-\lambda}\, d\lambda \quad . \tag{109}$$

The first integral is evaluated from the substitution $\lambda = \sigma\cos\theta$ and the well known formula of Glauert,

$$\frac{1}{\pi}\oint_0^{\pi} \frac{\cos r\theta \, d\theta}{\cos\theta - \cos\phi} = \frac{\sin r\phi}{\sin\phi} \tag{110}$$

and for $g_0(\lambda) = 1$ leads to

$$\frac{1}{\pi}\oint_{-\sigma}^{\eta} \frac{\sqrt{\sigma^2-\lambda^2}}{\nu-\lambda}\, d\lambda = \nu \quad . \tag{111}$$

The second integration is routine, and we obtain

$$I_0(\eta, \sigma) = -\sqrt{\sigma^2 - \eta^2} \quad . \tag{112}$$

The integration of I_0' leads to $J_0(\xi, \eta)$.

$$J_0(\xi, \eta) = -(\eta/2)\sqrt{\sigma^2 - \eta^2} - (\sigma^2/2) \sin^{-1}(\eta/\sigma) \quad . \tag{113}$$

In evaluating the integrals K_{00}, K_{10}, L_{00}, and L_{10} it is necessary to consider whether the integration is carried out in an interior or an edge region, i. e., whether η = constant or $\eta = \eta(\xi)$ during the ξ integration. For a delta wing of semiapex angle δ the semispan is given by $\sigma = \xi \tan \delta$, so that along the edge $\eta(\xi) = \sigma = \xi \tan \delta$. Carrying out the integration in the interior regions, and letting $\bar{\eta} = \eta \operatorname{ctn} \delta$ in the interior regions, yields

$$K_{00}(\xi, \eta) = \int f_0(\xi) \, J_0(\xi, \eta = \text{const}) \, d\xi \tag{114a}$$

$$= -\tan^2 \delta \left\{ (\xi \, \bar{\eta}/3)\sqrt{\xi^2 - \bar{\eta}^2} + (\xi^3/6) \sin^{-1}(\bar{\eta}/\xi) - (\bar{\eta}^3/6) \log (\xi + \sqrt{\xi^2 - \bar{\eta}^2}) \right\} \quad . \tag{114b}$$

$$K_{10}(\xi, \eta) = \int f_1(\xi) \, J_0(\xi, \eta = \text{const}) \, d\xi \tag{115a}$$

$$= -\tan^2 \delta \left[(1/24)(5\xi^2 \, \bar{\eta} - 2\bar{\eta}^3)\sqrt{\xi^2 - \eta^2} + (\xi^4/8) \sin^{-1}(\bar{\eta}/\xi) \right] \tag{115b}$$

$$L_{00}(\xi, \eta) = \int f_0'(\xi) \, J_0(\xi, \eta = \text{const}) \, d\xi \tag{116a}$$

$$= 0 \tag{116b}$$

$$L_{10}(\xi, \eta) = \int f_1'(\xi) \, J_0(\xi, \eta = \text{const}) \, d\xi \tag{117a}$$

$$= K_{00}(\xi, \eta) \quad . \tag{117b}$$

Carrying out the integration in the edge region yields

$$K_{00}(\xi,\eta) = \int f_0(\xi) J_0(\xi,\eta = \xi \tan \delta) d\xi \tag{118a}$$

$$= -(\pi/12)\xi^3 \tan^2 \delta \tag{118b}$$

$$K_{10}(\xi,\eta) = \int f_1(\xi) J_0(\xi,\eta = \xi \tan \delta) d\xi \tag{119a}$$

$$= -(\pi/16)\xi^4 \tan^2 \delta \tag{119b}$$

$$L_{00}(\xi,\eta) = \int f_0'(\xi) J_0(\xi,\eta = \xi \tan \delta) d\xi \tag{120a}$$

$$= 0 \tag{120b}$$

$$L_{10}(\xi,\eta) = \int f_1'(\xi) J_0(\xi,\eta = \xi \tan \delta) d\xi \tag{121a}$$

$$= K_{00}(\xi,\eta) \quad . \tag{121b}$$

Considering the right side of the wing as a single elemental area whose four corners are the apex (taken twice), the center line at the trailing edge and the wing tip, we may evaluate the coefficients M_{mn}, N_{mn}, and K_{mn} at the corners from which we may obtain the net coefficients $M_{mn}^{1,1}$, $N_{mn}^{1,1}$, and $K_{mn}^{1,1}$. For both modes considered, all three quantities vanish along the center line and the net coefficients become the values computed at the tip. We therefore obtain

$$M_{00}^{ij} = f_0'(1) J_0(1, \tan \delta) \tag{122a}$$

$$= 0 \tag{122b}$$

$$M_{10}^{ij} = f_1'(1) J_0(1, \tan \delta) \tag{123a}$$

$$= -(\pi/4) \tan^2 \delta \tag{123b}$$

$$N_{00}^{ij} = f_0(1) J_0(1, \tan\delta) + L_{00}(1, \tan\delta) \tag{124a}$$

$$= -(\pi/4) \tan^2 \delta \tag{124b}$$

$$N_{10}^{ij} \doteq f_1(1) J_0(1, \tan\delta) + L_{10}(1, \tan\delta) \tag{125a}$$

$$= -(\pi/3) \tan^2 \delta \tag{125b}$$

$$K_{00}^{ij} = -(\pi/12) \tan^2 \delta \tag{126}$$

$$K_{10}^{ij} = -(\pi/16) \tan^2 \delta \tag{127}$$

Hence, from Eq. (101) the total vertical force becomes

$$F(t) = 2\rho V^2 c^2 \Sigma\Sigma [M_{mn}^{ij} a_{mn} + N_{mn}^{ij} (c/V) \dot{a}_{mn} + K_{mn}^{ij} (c/V)^2 \ddot{a}_{mn}] \tag{128a}$$

$$= -2\rho V^2 c^2 \tan^2 \delta [(\pi/4)(c/V)(\dot{h}_o/c) + (\pi/12)(c/V)^2 (\ddot{h}_o/c)$$

$$+ (\pi/4)\alpha + (\pi/3)(c/V)\dot{\alpha} + (\pi/16)(c/V)^2 \ddot{\alpha}] \quad . \tag{128b}$$

In the case of harmonic pitching about station $x = \xi_0 c$, we let $h_0/c = -\xi_0 \alpha$ and we obtain the lift coefficient

$$C_Z = -(\pi AR/2)\, \alpha [1 + ik_c(4/3 - \xi_0) + k_c^2 (\xi_0/3 - 1/4)] \tag{129}$$

where $C_Z = F/(1/4)\rho V^2 c^2 \tan\delta$ and $AR = 4 \tan\delta$ and the reduced frequency is based on the root chord, $k_c = \omega c/V$. This result agrees with that given by Miles [Ref. 38, Eq.(11), p. 357]. The above digression has illustrated the evaluation of the various integrals. A complete evaluation of the integrals for the ranges $m = 0(1)4$ and $n = 0(2)8$ for the symmetrical case, and $n = 1(2)9$ for

the antisymmetrical case, is given in Ref. 49, which provides sufficient data for the analysis of 15 control points, e.g., for the planform divided as shown in Fig. 7.

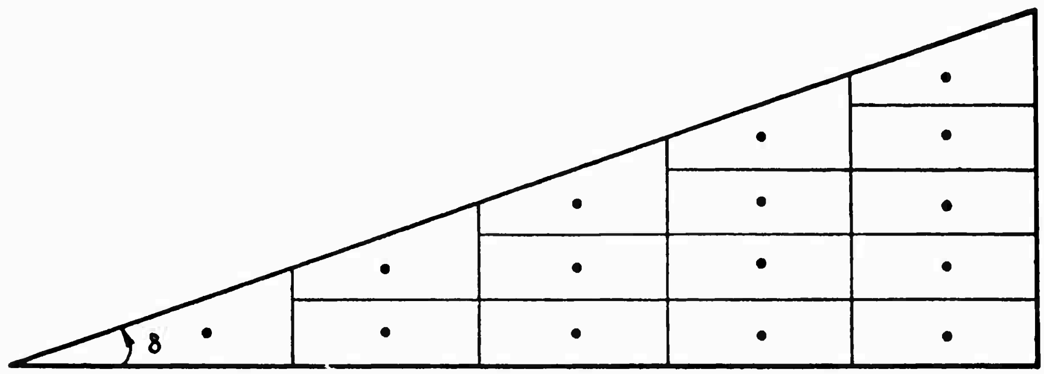

Fig. 7. Division of Delta Wing for 15 Control Points

To complete the derivation of the AICs the matrix counterpart of Eq. (101) is

$$\left\{F(x_i, y_j, t)\right\} = 2\rho V^2 c^2 \Big([M^{ij}_{mn}]\left\{a_{mn}\right\} + (c/V)\,[N^{ij}_{mn}]\left\{\dot{a}_{mn}\right\} + (c/V)^2\,[K^{ij}_{mn}]\left\{\ddot{a}_{mn}\right\}\Big) \tag{130}$$

where the dependence of a_{mn} on real time is now implied. The matrix counterpart of Eq. (86) is written

$$h(x_i, y_j, t) = c[H^{ij}_{mn}]\left\{a_{mn}\right\} \tag{131}$$

where $H^{ij}_{mn} = f_m(\xi_i)\, g_n(\xi_j)$. The solution for $\left\{a_{mn}\right\}$ in terms of $\left\{h\right\}$ may follow directly from inverting Eq. (120) but for a number of numerical reasons a least squares solution is preferable. The first is that a large matrix of polynomials, e.g., $[H^{ij}_{mn}] = [\xi^m \eta^n]$, tends to be singular. The second is that a Lagrangian surface fit, as we originally proposed, tends to be poorly behaved between collocation points. Finally, it is often necessary to consider a larger

number of structural degrees of freedom in a vibration analysis than it is necessary to consider aerodynamic degrees of freedom in a flutter analysis. For these reasons we consider Eq. (131) to be over-determined, i. e., the set $\{h\}$ is larger than the set $\{a_{mn}\}$, and the solution to Eq. (131) only exists in the statistical sense. This least squares solution is

$$\{a_{mn}\} = (1/c)\left([H^{ij}_{mn}]^* [\Delta S_{ij}] [H^{ij}_{mn}]\right)^{-1} [H^{ij}_{mn}]^* [\Delta S_{ij}] \{h\} \tag{132}$$

where the diagonal weighting matrix of elemental areas, $[\Delta S_{ij}]$, arises from minimizing the mean square error over the surface area. The oscillatory AICs follow from substituting Eq. (132) into Eq. (130), assuming harmonic motion, and comparing the result with Eq. (2). We obtain

$$[C_h] = (2/k_r^2)(c/s)[B^{ij}_{mn}]\left([H^{ij}_{mn}]^* [\Delta S_{ij}][H^{ij}_{mn}]\right)^{-1} [H^{ij}_{mn}]^* [\Delta S_{ij}] \tag{133}$$

where

$$[B^{ij}_{mn}] = [M^{ij}_{mn}] + ik_r(c/b_r)[N^{ij}_{mn}] - k_r^2(c^2/b_r^2)[K^{ij}_{mn}] \tag{134}$$

and the choice of reference semichord remains arbitrary.

The theoretical limitations on the applicability of slender-wing theory are outlined by Miles.[38] The Mach number limitation is given by

$$|M^2 - 1| << (AR)^{-2} \quad , \tag{135}$$

and the reduced frequency limitation is given by

$$k_r(c/b_r) << (MAR)^{-2} \quad . \tag{136}$$

The practical significance of these limitations must be determined experimentally to determine a meaningful boundary between low-and high-aspect ratio. Some experimental correlation has been obtained on a very low-aspect ratio rectangular wing by Fralich and Hedgepeth.[51]

3. Supersonic Lifting Surface Theory

a. Introduction

For many applications involving low-aspect ratio planforms, the low supersonic regime with subsonic leading edges is next in importance after the high subsonic and transonic regime. For the purposes of achieving a unified format for calculating the AIC matrices by the relations of Eqs. (35) and (36), we choose the supersonic kernel function procedure, based on acceleration potential as developed by Cunningham, Watkins, and Woolston[52] based on earlier work of Watkins and Berman.[53] (See Ref. 38 for a thorough treatment of the various methods of solving supersonic lifting surface problems.) We shall now review some work in possible alternative methods and indicate what has been done to develop the AIC matrices and how this development is influenced by the type of aerodynamic solution.

We classify the solutions of interest in aeroelasticity as follows:

Class 1: Closed form velocity potential solutions for specific analytic forms of downwash variation;

Class 2: "Box" method numerical solutions for velocity potential distribution;

Class 3: Pressure loading or acceleration potential kernel function solutions.

b. Discussion of the Methods

In Classes 1 and 2, the integral equation has been inverted explicitly to obtain the velocity potential in terms of the downwash. It has been shown (e.g., Refs. 34 and 38) that for the harmonic case $[w = Re(-\bar{w}\exp(ikt))]$

$$\phi(x, y, 0;k) = \frac{1}{\pi} \int_{S_{M.f.c.}} w(\xi, \eta) \exp\left[- \frac{ikM^2(x-\xi)}{\beta^2} \right] \frac{\cos\left(\frac{kM}{\beta^2} \sqrt{(x-\xi)^2 - \beta^2 (y-\eta)^2} \right)}{\sqrt{(x-\xi)^2 - \beta^2 (y-\eta)^2}} \, dS(\xi, \eta) \tag{137}$$

For the so-called "simple planform" (e.g., with supersonic edges) the solution is complete because the Mach forecone area of (x, y), $S_{M.f.c.}$, lies completely on the planform. For subsonic edges, the problem is complicated by the fact that $w(\xi, \eta)$ is unknown in the region between leading edges and the Mach cone emanating from the wing apex (see Fig. 8). This region is called the "diaphragm" region for which the boundary condition is the vanishing of the potential. The box method solutions of Class 2 proceed with Eq. (137) formally, and compute the diaphragm downwashes at a discrete set of points from the vanishing of the potential condition; the reader is referred to Pines, Dugundji, and Neuringer,[54] Zartarian and Hsu,[55] Zartarian,[56] and Li[57] for details. Further developments have been made for computation by Ashley and Andrew.[58] The numerous static versions (see Etkin and Woodward,[10] for example) will not be discussed specifically, but have been useful in problems of optimum camber distribution. We consider the prospects for box solutions in the collocation AIC format briefly.

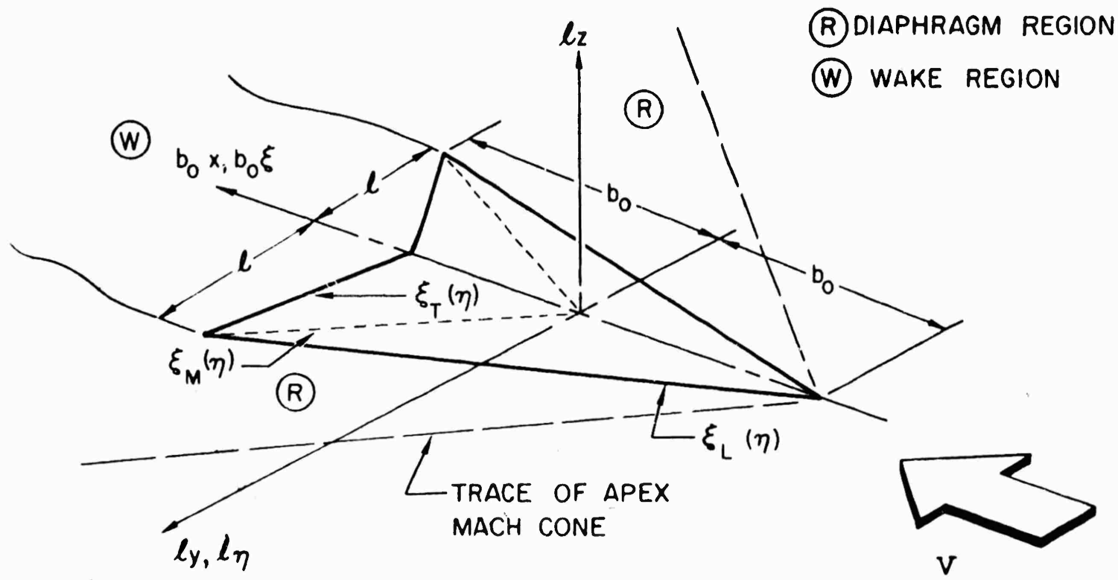

Fig. 8. Supersonic Planform Geometry

To begin with, an essential point with the collocation format is the lumping of loads to obtain the force-pressure relationship. The box method assumes constant pressure and downwash on each "box" so that such an element is necessarily a much smaller subdivision of the planform than is the ΔS_j associated with collocation control point force, wherein we are seeking the coarsest possible grid consistent with a good representation of natural vibration modes shapes and frequencies. Thus, unless chordwise deformation is a crucial feature of the aeroelastic problem in question, we shall seek the smallest possible number of aerodynamic control points. We are thus led to the notion that a rather large number of boxes must be subassociated with a collocation point such that

$$F_j = \sum_{\nu=1}^{\nu_j} \Delta S_{j\nu} \, \Delta p_\nu \quad . \tag{138}$$

Now Eq. (138) presents no difficulties for the fixed box grid; however, the consensus experience with the box method indicates the superiority of the Mach box approach,[55, 56] wherein the spanwise dimension of the box is b_1/β and the chordwise dimension is b_1. This gives the same number of boxes covering the entire diaphragm region for all Mach numbers and is computationally simpler to handle. On the other hand, as $M \to 1, \beta \to 0$, the lateral box dimension becomes prohibitively large unless the basic grid is proportionately finer (in which case the computing time becomes excessive). Notwithstanding this, the solution has become a practical one as generally applied to the computation of modal generalized aerodynamic forces, for say $M > 1.3$. Now regarding the collocation method, the variable location of Mach box grid presents a great difficulty in the logic of assigning boxes to a fixed grid control point so that we may apply Eq. (138). Furthermore, to satisfy the boundary conditions of the

aerodynamic problem (especially $\Delta p = 0$ on R), the kernel function approach is superior. In view of the importance of the subsonic edge regime, the deterioration of mechanical accuracy of the box method as $M \rightarrow 1$, and the "floating grid" logic problem, we turn our attention to the methods of Class 3.

It is desirable to have a unified subsonic-transonic-supersonic lifting surface approach to the development of AICs. For this purpose, the pressure kernel function approach (Class 3 solution) enables use of much of the computer coding logic developed for the subsonic program. In particular, the $[W_{ij}]$ is the same, as is the relation of $[A^i_{nm}]$ to $[D^i_{nm}]$ [Eq. (47b)]. As will be seen, only the kernel, the loading functions, and the domain of integration ($S_{M.f.c.}$ instead of the entire planform) are changed; hence, the $[B^i_{nm}]$ and $[D^i_{nm}]$ matrices are the only ones changed. Furthermore, if the moments of the chordwise pressure loadings used in the $[B^i_{nm}]$ development employing the constrained minimum vector length solution [See Eqs. (71) and (72)] are developed by a numerical quadrature technique, then the same $[B^i_{nm}]$ matrix program may be retained! For a large number of chordwise loads, we require additional conditions of the form*

$$\sum_{j=1}^{N_j(y)} B^j_{nm}\, h(x_j) = \int_{y-\Delta y/2}^{y+\Delta y/2} dy \int_{\xi_{\ell.e.}}^{\xi_{t.e.}} \frac{\Delta p_{nm}}{\pi \rho V^2 \ell^2}\, h(\xi, y)\, d\xi \qquad (139)$$

where the h(x) represent deflection functions whose generalized forces are to be preserved by the equivalent set of concentrated loads in the AIC representation. The above approach is the essence of the current line of development of AICs.[59]

The remaining Class 1 or closed form solutions, will now be discussed. In this category are the works of Watkins and Berman[60] and Miles[38] which are based on low reduced frequency solutions. Ref. 60 considers a cubic frequency expansion for a wing whose deformation is of the form

*These functions may be any functions representative of the type of mode shapes thought to be important in the intended application. We thus find some kinship to classical modal aeroelastic analysis here!

$$h(x, y) = \sum_{r=0}^{2} \sum_{s=0}^{2} d_{rs} x^r y^s \quad . \tag{140}$$

Miles (Ref. 38, Sections 4.4 and 4.5) illustrates a procedure whereby the unsteady solution is expressed as a power series in $K = kM/\beta$. If, following Miles,

$$\phi(x/\beta, y, z, \beta t) = \bar{\phi}(x/\beta, y, z) e^{ikMt} \tag{141a}$$

$$\bar{\varphi} = \Phi(x/\beta, y, z) \exp(ikM^2 x/\beta^2) \tag{141b}$$

wherein the coordinate and time are made dimensionless by reference to ℓ and ℓ/V, respectively.

Then, a solution is found of the form

$$\Phi = \sum_{n=0}^{\infty} K^{2n} (\Phi_{2n} + iK \Phi_{2n+1}) \tag{142}$$

For the low frequency case the solution of Φ is given in terms of Φ_0, the steady state term (See Ref. 38, Eq. 4.5.2). The procedure can be continued for the higher Φ_n. As a starting point, the solutions of Lance[61] provide steady state solutions for a general cubic $[r = 0(1)3,\ s = 0(1)3]$ in Eq. (140) above. Rodden and Revell[62] have developed AICs for narrow delta wings utilizing these results. The course of this development follows closely the slender-wing development described in Section II.B.2, and the loading functions can be expressed as a linear combination of the loadings of the type found in slender-wing theory. The coefficients are Mach number dependent, involving elliptic integral functions.

The solutions of the Class 1 type have the difficulty that the deflection interpolation in terms of Eq. (132) "in the large" can be inadequate for low-aspect ratio wings with extensive chordwise deformation. However, the result of Eq. (132) can be improved if N_j, the number of deflections, exceeds greatly the number of undetermined coefficients in Eq. (140).

Before concluding, we shall mention some alternative developments of the supersonic kernel function procedure. Ferguson and Guderly[63] propose a solution by deriving a Volterra integral equation of the second kind, whose domain of integration is confined to the planform and, as such, provides an extension of Evvard's method[9] to unsteady flow for the case of subsonic side edges and a supersonic trailing edge; however, the iterative procedure proposed to solve the integral equation is not considered as simple as the direct matrix solution of the Watkins' acceleration potential kernel function method. (For a review of earlier difficulties in attempting to apply Evvard's area cancelling techniques to unsteady flow, see Ref. 38, pp. 71-72.) We note finally that Stanišić,[64] in an apparently independent series of investigations, has advocated the use of the acceleration potential-kernel function approach.

To summarize briefly, the supersonic acceleration potential method is considered to be the most efficient for the development of AICs as defined here for the following reasons:

(1) It automatically satisfies the pressure boundary conditions in the diaphragm region R (see Fig. 8);

(2) It utilizes much of the same digital computing logic as does the subsonic problem providing a more nearly unified basis for lifting surface theory over the range of Mach numbers;

(3) It is versatile for handling arbitrary planforms and interference effects over the entire range of reduced frequency; hence, it is suitable for practical engineering use where rapid assessment of a variety of aerodynamic planforms is an important capability in the earlier phases of design;

(4) It is not enslaved to a particular form of surface deformation function (as with the Class 1 solutions);

(5) It does not suffer additional deterioration of accuracy(beyond the implicit limits of linear potential theory) which are found in the Class 2 solutions as $M \rightarrow 1$;

(6) It is satisfactory for the supersonic edge case (notwithstanding the possibility of an explicit solution in this case); hence, the method can serve as a logical basis for a unified supersonic lifting surface theory for development of AICs. (Further results of Ref. 59 will be published subsequently.)

4. Hypersonic Lifting Surface Theory

a. Introduction

In the hypersonic Mach number regime, our first consideration will be the development of AICs based on piston theory. The later considerations will discuss the various classes of approximations whose validity depends upon the hypersonic similarity parameter $M\theta_b$, where θ_b is the effective local body surface inclination to the free stream with due account of angle of attack and viscous displacement effects.

b. AICs by Piston Theory

The piston theory has been mentioned as a strip theory in Section II. A. However, as a point pressure-downwash relationship, it is more appropriately considered as a lifting surface theory. As such, the pressure distribution can be found for any arbitrary surface deflection mode, from which the AICs can be derived. However, this solution has not been obtained as yet in general form, and results will be shown here only for the case of a rigid chord with a control surface. The derivation of the oscillatory aerodynamic coefficients and the AICs is given in Ref. 65 and will not be reproduced here. It differs only slightly from the treatment of Ashley and Zartarian,[25] in that the third-order pressure coefficient is generalized to account for sweep and steady angle of attack, and, following a suggestion of Morgan, Huckel, and Runyan,[66] to conform with the second-order quasi-steady theory of Van Dyke.[67] Therefore, the generalized pressure equation is written in the form

$$p - p_0 = \rho_0 a_0^2 \left[C_1(w/a_0) + C_2(w/a_0)^2 + C_3(w/a_0)^3 \right] \quad . \tag{143}$$

where, for the usual piston theory,

$$L_{h0} = -iK_1/k \tag{146a}$$

$$L_{\alpha 0} = -K_1/k^2 - iK_2/k \tag{146b}$$

$$L_{\beta 0} = -K_4/k^2 - i(K_5 - 2K_4\xi_h)/k \tag{146c}$$

$$M_{h0} = -iK_2/k \tag{146d}$$

$$M_{\alpha 0} = -K_2/k^2 - iK_3/k \tag{146e}$$

$$M_{\beta 0} = -K_5/k^2 - i(K_6 - 2K_5\xi_h)/k \tag{146f}$$

$$T_{h0} = -i(K_5 - 2K_4\xi_h)/k \tag{146g}$$

$$T_{\alpha 0} = -(K_5 - 2K_4\xi_h)/k^2 - i(K_6 - 2K_5\xi_h)/k \tag{146h}$$

$$T_{\beta 0} = -(K_5 - 2K_4\xi_h)/k^2 - i(K_6 - 4K_5\xi_h + 4K_4\xi_h^{\,2})/k \tag{146i}$$

where

$$\xi_h = x_h/2b \tag{147}$$

$$K_1 = (1/M)\left[C_1 + 2C_2MI_1 + 3C_3M^2(I_4 + \alpha_0^{\,2})\right] \tag{148a}$$

$$K_2 = (1/M)\left[C_1 + 4C_2MI_2 + 3C_3M^2(2I_5 + \alpha_0^{\,2})\right] \tag{148b}$$

$$K_3 = (4/3M)\left[C_1 + 6C_2MI_3 + 3C_3M^2(3I_6 + \alpha_0^{\,2})\right] \tag{148c}$$

$$K_4 = (1/M)\left\{C_1(1 - \xi_h) + 2C_2MJ_1 + 3C_3M^2\left[J_4 + \alpha_0^{\,2}(1 - \xi_h)\right]\right\} \tag{148d}$$

$$C_1 = 1 \tag{144a}$$

$$C_2 = (\gamma + 1)/4 \tag{144b}$$

$$C_3 = (\gamma + 1)/12 \tag{144c}$$

and, for the quasi-steady theory,

$$C_1 = M/\sqrt{M^2 - \sec^2 \Lambda} \tag{145a}$$

$$C_2 = \frac{M^4(\gamma + 1) - 4 \sec^2\Lambda (M^2 - \sec^2 \Lambda)}{4(M^2 - \sec^2 \Lambda)^2} \tag{145b}$$

$$C_3 = (\gamma + 1)/12 \tag{145c}$$

where the value of C_3 is taken from piston theory since Van Dyke gives only a second-order solution. Ashley and Zartarian[25] have shown that the piston theory is applicable if any of the conditions $M^2 \gg 1$, $Mk \gg 1$, or $k^2 \gg 1$ are met. We see that if the reduced frequency is low, the Mach number must necessarily be high; however, if the reduced frequency is large, the Mach number need not necessarily be large, and, in fact, it could be transonic or even subsonic. From this it is apparent that any correction introduced to bring piston theory into line with linearized supersonic theory must be considered as a low frequency approximation. The use of Eq. (145) extends the lower Mach number limit of piston theory at low frequencies. Eq. (145) is seen to be the more general result, since if sec Λ is taken to be zero, Eqs. (144) result, and no correction is made.

The derivation of the oscillatory aerodynamic coefficients referred to the leading edge (i.e., both motion and pitching moment) leads to the following values:

$$L_{h0} = -iK_1/k \tag{146a}$$

$$L_{\alpha 0} = -K_1/k^2 - iK_2/k \tag{146b}$$

$$L_{\beta 0} = -K_4/k^2 - i(K_5 - 2K_4\xi_h)/k \tag{146c}$$

$$M_{h0} = -iK_2/k \tag{146d}$$

$$M_{\alpha 0} = -K_2/k^2 - iK_3/k \tag{146e}$$

$$M_{\beta 0} = -K_5/k^2 - i(K_6 - 2K_5\xi_h)/k \tag{146f}$$

$$T_{h0} = -i(K_5 - 2K_4\xi_h)/k \tag{146g}$$

$$T_{\alpha 0} = -(K_5 - 2K_4\xi_h)/k^2 - i(K_6 - 2K_5\xi_h)/k \tag{146h}$$

$$T_{\beta 0} = -(K_5 - 2K_4\xi_h)/k^2 - i(K_6 - 4K_5\xi_h + 4K_4\xi_h^2)/k \tag{146i}$$

where

$$\xi_h = x_h/2b \tag{147}$$

$$K_1 = (1/M)\left[C_1 + 2C_2MI_1 + 3C_3M^2(I_4 + \alpha_0^2)\right] \tag{148a}$$

$$K_2 = (1/M)\left[C_1 + 4C_2MI_2 + 3C_3M^2(2I_5 + \alpha_0^2)\right] \tag{148b}$$

$$K_3 = (4/3M)\left[C_1 + 6C_2MI_3 + 3C_3M^2(3I_6 + \alpha_0^2)\right] \tag{148c}$$

$$K_4 = (1/M)\left\{C_1(1 - \xi_h) + 2C_2MJ_1 + 3C_3M^2\left[J_4 + \alpha_0^2(1 - \xi_h)\right]\right\} \tag{148d}$$

$$K_5 = (1/M)\left\{C_1(1 - \xi_h^2) + 4C_2MJ_2 + 3C_3M^2\left[2J_5 + \alpha_0^2(1 - \xi_h^2)\right]\right\} \quad (148e)$$

$$K_6 = (4/3M)\left\{C_1(1 - \xi_h^3) + 6C_2MJ_3 + 3C_3M^2\left[3J_6 + \alpha_0^2(1 - \xi_h^3)\right]\right\}, \quad (148f)$$

and the thickness integrals for the airfoil having the symmetrical thickness distribution $2g(x)$ are

$$I_n = \begin{cases} (1/2b)^n \int_0^{2b} x^{n-1}\left(\frac{dg}{dx}\right)dx, & \text{for } n = 1,2,3 \\ (1/2b)^{n-3} \int_0^{2b} x^{n-4}\left(\frac{dg}{dx}\right)^2 dx, & \text{for } n = 4,5,6 \end{cases} \quad (149)$$

$$J_n = \begin{cases} (1/2b)^n \int_{x_h}^{2b} x^{n-1}\left(\frac{dg}{dx}\right)dx, & \text{for } n = 1,2,3 \\ (1/2b)^{n-3} \int_{x_h}^{2b} x^{n-4}\left(\frac{dg}{dx}\right)^2 dx, & \text{for } n = 4,5,6 \end{cases} \quad (150)$$

The thickness integrals are evaluated in Ref. 65 for an airfoil cross section idealized into a parabola from the (sharp) leading edge to the point of maximum thickness, another parabola to the control surface hinge line, and a straight line to the (possibly blunt) trailing edge.

The development of the AICs in Section II. A assumed the oscillatory coefficients were referred to the quarter-chord. Since the present coefficients are referred to the leading edge, an alternative derivation of the AICs for the strip is necessary. This is also given in Ref. 65, assuming again the control points to be at the quarter-chord, hinge line, and trailing edge, and the resulting expression is

$$\left[C_h\right] = 4(b/b_r)^2(\Delta y/s)\begin{bmatrix} (1 + b/2d) & -b/d & (b/c_a)(3b/2d-1) \\ -b/2d & b/d & -(b/c_a)(3b/2d) \\ 0 & 0 & b/c_a \end{bmatrix}$$

$$\times \begin{bmatrix} L_{h0} & L_{\alpha 0} & L_{\beta 0} \\ M_{h0} & M_{\alpha 0} & M_{\beta 0} \\ T_{h0} & T_{\alpha 0} & T_{\beta 0} \end{bmatrix}\begin{bmatrix} (1 + b/2d) & -b/2d & 0 \\ -b/d & b/d & 0 \\ b/d & -(b/d+b/c_a) & b/c_a \end{bmatrix} \quad (151)$$

which in the absence of a control surface reduces to

$$\left[C_h\right] = 4(b/b_r)^2(\Delta y/s)\begin{bmatrix} (1 + b/2d) & -b/d \\ -b/2d & b/d \end{bmatrix}\begin{bmatrix} L_{h0} & L_{\alpha 0} \\ M_{h0} & M_{\alpha 0} \end{bmatrix}\begin{bmatrix} (1+b/2d) & -b/2d \\ -b/d & b/d \end{bmatrix}. \quad (152)$$

We have illustrated a technique to extend the lower Mach number limit of piston theory. The question remains as to the upper Mach number limit. This question has been discussed by Morgan, Runyan, and Huckel,[68] and by Miles.[69] Experimental correlation has been obtained up to M = 6.86. Miles[69] argues that the basic philosophy of piston theory is valid if changes in entropy associated with the unsteady flow can be neglected. Then the perturbation pressure p' may be calculated from the plane-wave relation

$$p' = \rho a w' \quad (153)$$

where ρ and a denote the local density and sonic speed for the <u>steady</u> flow, and w' is the unsteady component of the downwash velocity. It is emphasized that ρ and a need not be based on the assumption of an ideal fluid. In short, Eq. (153) states that the unsteady solution is known when the steady one is obtained, provided the changes in entropy due to unsteadiness may be neglected. Miles suggests procedures for calculating ρ and a when the usual piston theory [i.e., Eq. (143)] becomes inadequate. Morgan, Runyan, and Huckel[68] suggest that a probable upper limit of validity of piston theory is about Mach 15. Beyond this, the problems of entropy changes, dissociation, and boundary layer interaction must be considered.

c. General Hypersonic Inviscid Considerations

In the hypersonic regime, a given blunt-nosed slender body or wing can be thought of as possessing several distinct local flow regimes depending on the local hypersonic similarity parameters (see Ref. 70, Sec. 1. 3).

$$K_{\theta_b} = M\theta_b(x) \tag{154}$$

$$K_\alpha = M\alpha(t) \tag{155}$$

where $\theta_b(x)$ is the local streamwise slope of the body. If we specialize to the case of small unsteady motion, we may write for longitudinal motion

$$\alpha = \alpha_0 + \dot{h}(x,t)/V + \dot{\alpha}(x-x_0) \tag{156}$$

where

$$\left|\dot{h}/V\right|, \left|\dot{\alpha}\ell/V\right| << 1 \tag{157}$$

Then in this case $K_\alpha \simeq K_{\alpha_0}$ and becomes fixed for a given steady state vehicle attitude α_0. Now for any blunt-nosed vehicle θ_b varies from 90 degrees to nearly zero. We must also add $d\,\delta^*/dx$, the boundary layer displacement thickness gradient, to the slope, and thus the effective local streamwise slope, $\theta_b + d\delta^*/dx$, is always finite.

We now consider a typical hypersonic wing section as shown in Figure 9. For convenience, we note that Hayes and Probstein classify the hypersonic regimes as follows:

(1) Thin shocklayer or "strong shock" (nose) regime

$$M\theta_b(x) >> 1 \tag{158}$$

(2) Small density ratio regime
($\epsilon = \rho_0/\rho_2 = u_{2n}/u_{0n}$)

$$\epsilon << 1 \tag{159}$$

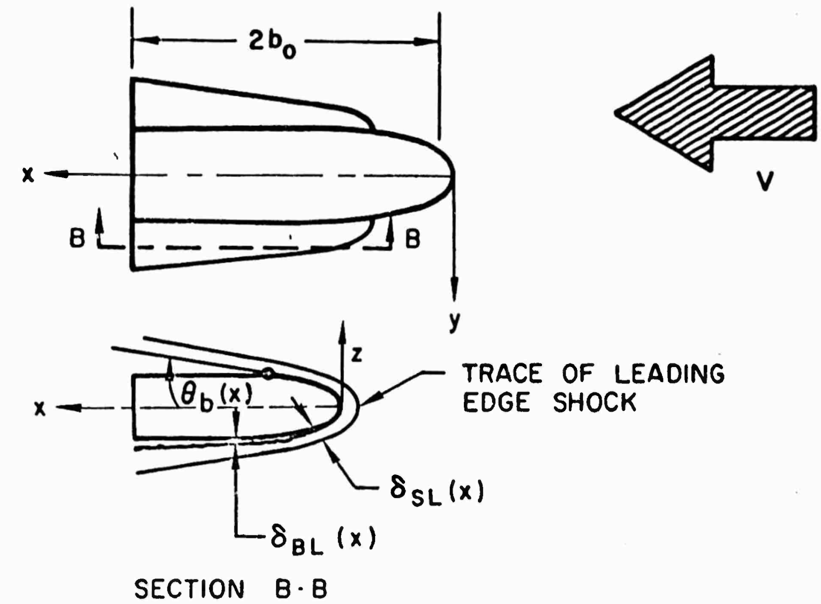

Fig. 9. Hypersonic Planform and Airfoil Section

(3) Slender body regime

$$\theta_b(x) \ll 1 \tag{160}$$

(4) Linear regime

$$M\theta_b(x) \ll 1 \tag{161}$$

In the nose regime, the simplest approach for aeroelastic purposes is the Newtonian impact theory. Applications of Newtonian theory to unsteady flow have been given by Hayes and Probstein,[70] Kennet,[71] and Tobak and Wehrend.[72] A typical steady flow application is given by Jackson.[73] Not all thin shock layer problems are adequately described by Newtonian laws. Kennet[71] has studied unsteady perturbations in the stagnation point neighborhood within the shock layer due to plunging oscillations, and shows some severe departures from quasi-steady Newtonian theory at $M = 5.8$. In this approach, he derives a third-order linear partial differential equation for $\bar{u}$, the unsteady meridian tangential velocity component in the shock layer, (by manipulation of the momentum and continuity equations), which he solves by a power series in $\eta = y/\delta_{SL}$ (see Figure 10). He finds shock oscillation amplitudes related

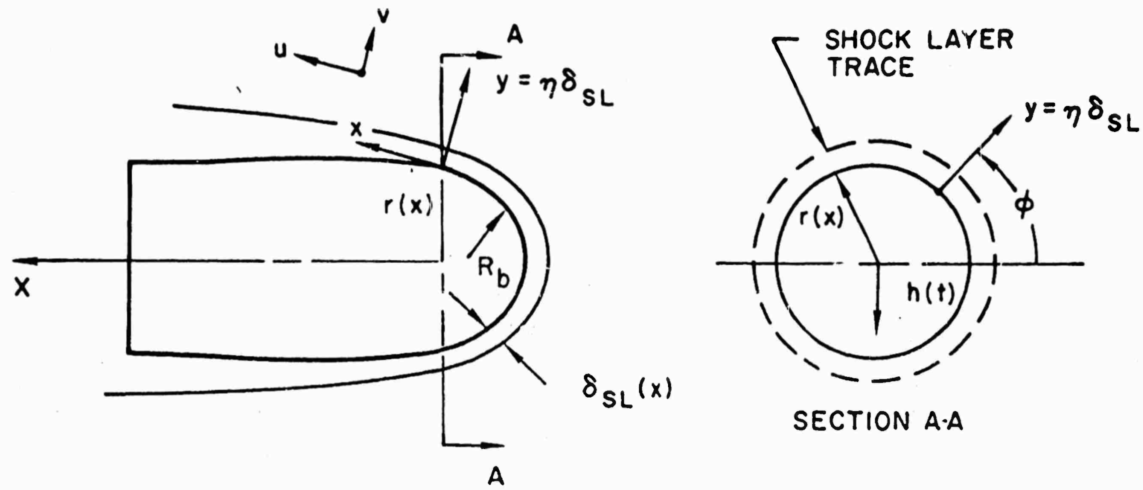

Fig. 10. Blunt Body Shock Layer Geometry

to the plunging motion and surface pressure derivatives with respect to h, $\dot{h}$, and $\ddot{h}$. This method requires knowledge of the steady flow within the shock layer, which is rotational and is nonuniform, and requires satisfaction of the Rankine-Hugoniot relations at the bow shock. These features are similar to those discussed with respect to the transonic problem; however, here the vorticity is of the essence in describing the flow field inside the shock layer. From the point of view of the aeroelastician, interest centers on the transverse loading and therefore on the loading on the part of the body where $\theta_b << 1$, but where still $M\theta_b$ is large for high hypersonic speeds. For this situation, Newtonian theory should be useful, and the stagnation point difficulties present no practical problem since these pressure fluctuations are directed chordwise, against a rather solid nose.

d. Newtonian Theory

The Newtonian pressure coefficient is given by

$$C_p = 2v_n^2/V^2 \quad . \tag{162}$$

Let the relative fluid velocity be given by

$$\vec{v} = \underset{\rightarrow}{i}V - \underset{\rightarrow}{k}w_b \quad . \tag{163}$$

The normal to the body for two-dimensional bodies is

$$\vec{n}_{u,\ell} = -\underset{\rightarrow}{i} \sin \theta_b(x) \pm \underset{\rightarrow}{k} \cos \theta_b(x) \tag{164a}$$

and for axisymmetric bodies (see Figure 11) is

$$\vec{n} = -\underset{\rightarrow}{i} \sin \theta_b(x) + \underset{\rightarrow}{i} \cos \theta_b(x) \cos \theta + \underset{\rightarrow}{k} \cos \theta_b(x) \sin \theta \quad . \tag{164b}$$

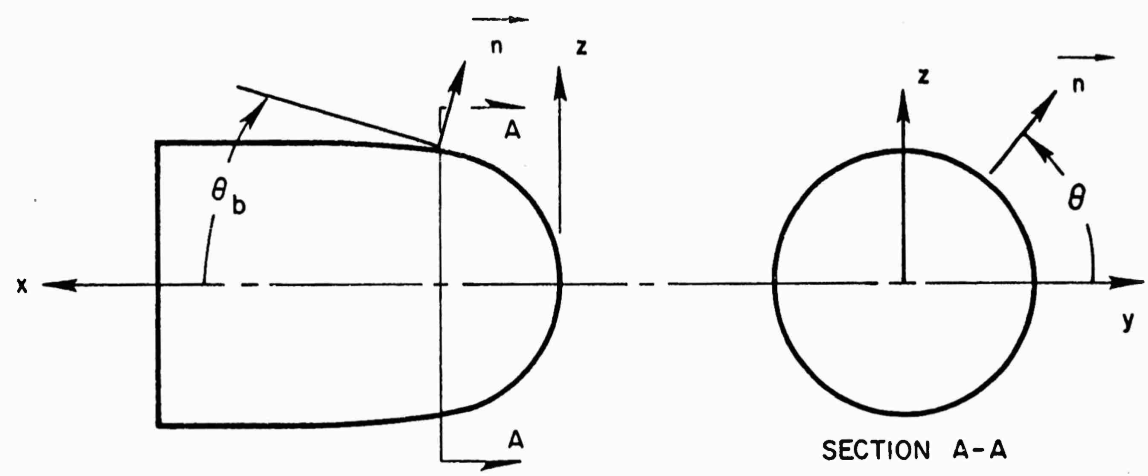

Fig. 11. Orientation of the Unit Normal Vector

Then for two-dimensional flow, the lower and upper surface pressure coefficients are given by

$$C_{p_{u,\ell}} = 2\left[\sin^2\theta_b \pm (w_b/V) \sin 2\theta_b + (w_b/V)^2 \cos^2\theta_b\right] \quad . \tag{165a}$$

For bodies of revolution we have

$$C_p = 2\left[\sin^2\theta_b + (w_b/V)\sin 2\theta_b \sin\theta + (w_b/V)^2\cos^2\theta_b \sin^2\theta\right] \quad . \tag{165b}$$

We have the requirement of positive pressure for all Newtonian theories

$$C_p \geq 0 \quad . \tag{166}$$

For practical purposes, Newtonian theory can be improved by using Lees'[74] empirical suggestion and replacing the factor 2 by the stagnation point pressure coefficient as is done in steady flow. This empirical form is simpler to use than the more logically appealing Busemann correction for centrifugal force which unfortunately can lead to negative pressures (see Ref. 70, Ch. 3).

e. Recent Methods

An alternative thin shock layer approach has its roots in "blast wave theory." Recently Zartarian, Hsu, and Ashley[75] have developed a variational approach for the equivalent, two-dimensional piston problem in the plane transverse to the body axis. This theory is aimed at the neighborhood $M\theta_b \simeq 1$ to supplement their extension of the well-known shock expansion theory to unsteady flow based on the quasi-steady assumption. Because many hypersonic, unsteady aeroelastic phenomena will occur at very low reduced frequencies, the quasi-static analyses appear more attractive than at transonic speeds. In this category, the shock expansion method as extended by Zartarian, Hsu, and Ashley is especially pertinent, for the effects of the nose shock can be handled quite easily. These investigators find distinct improvements over the widely used piston theory for the range $M\theta_b > 0.3$ although piston theory is fairly adequate up to $M\theta_b \simeq 0.7$.

The above-mentioned theories all have the simplicity of a point relation between pressure and downwash, and offer few difficulties to aeroelastic applications. All of the theories show the importance of thickness effects

which, as previously mentioned, should include the viscous displacement thickness δ^*. We next consider the viscous interaction problem.

f. Viscous Interactions

The problem of viscous interactions has received great attention in steady hypersonic flow theory, since aerodynamic control considerations are concerned with accurate predictions for Mach numbers and altitudes much higher than those at which maximum dynamic pressures are expected. In these cases of low Reynolds numbers the problems are very serious, and we find, for example, that the concept of a sharp leading edge is completely illusory; we have at least a "half-power" body since $\delta^*/x \sim (\gamma-1)M^2/\sqrt{Re_x}$. Now at very high Mach numbers $\delta_{SL} \sim \epsilon d_{l.e.}$ and $(\delta_{BL})_{st} \sim \sqrt{\nu d_{l.e.}/V}$, and the shock layer and boundary layer tend to merge at low Reynolds' numbers. However, if the leading edge diameter Reynolds number Re_d is sufficiently large, then the boundary layer thickness δ_{BL} is much less than the shock layer thickness δ_{SL}, and the inviscid flow dominates the stagnation region. According to Lees,[76] this condition is met if

$$Re_d = \rho_0 Vd/\mu_0 \geq 10^3 \quad . \tag{167}$$

Under such circumstances, the analysis of the type conducted by Kennet is adequate for the stagnation point vicinity.

We have already seen the importance of thickness on the pressure distribution in the treatment of piston theory; we consider therefore, following Zartarian, Hsu, and Ashley,[75] that to the first approximation the addition of the displacement thickness δ^* accounts sufficiently for the effects of viscosity. This factor becomes important at very high Mach numbers since the laminar boundary layer contribution to $M\theta_b$ is

$$\Delta(M\theta_b) = M\frac{d\delta^*}{dx} \sim \frac{(\gamma-1)M^3}{\sqrt{Re_x}} \sim \chi \tag{168}$$

where χ is the viscous interaction parameter.[70] It is well to recognize that the calculation of displacement thickness at high Mach number requires some familiarity with the field of aerothermodynamics. We therefore illustrate the necessary formulas with example calculations. The aeroelastician is primarily concerned with high dynamic pressure, which keeps the considerations well within the valid regime of continuum boundary layer theory (say $Re_x > 10^4$, see Ref. 70, Ch. 10, or Ref. 40, Ch. 14). Let us express Re_x in terms of dynamic pressure and free stream Mach number.

$$Re_x = \rho_0 Vx/\mu_0 \tag{169a}$$

$$= (1/2)\rho_0 V^2 (2x/Ma_0\mu_0) \tag{169b}$$

$$= q(2x/Ma_0\mu_0) \tag{169c}$$

where the zero subscript refers to the free stream condition. For instance, if $\mu_0 = 4 \times 10^{-7}$ lb-sec/sq ft, $a_0 = 10^3$ fps, and $q = 10^3$ psf, then $Re_x = 5 \times 10^6\,(x/M)$, which for $M = 10$, $x = 1$ ft, yields $Re_x = 5 \times 10^5$. Since transition Reynolds' numbers are of this order (see Ref. 77, p. 438), much of the boundary layer will be turbulent.

For a flat plate, the displacement thickness is given by (Ref. 77, p. 537)

$$\delta^* = (\delta^*/\theta)(C_F/C_{Fi})(C_{Fi}x/2) \tag{170}$$

where the momentum thickness θ is

$$\theta = (C_F/C_{Fi})(C_{Fi}x/2) \quad . \tag{171}$$

As an example, we consider values for C_F/C_{Fi} and C_{Fi} based on experimental results. Taking $Re_x = 10^6$ and $C_{Fi} = 0.003$ (see Ref. 77, p. 538), and using the values of $H = \delta^*/\theta$ given by Tucker[78] for a one-seventh power boundary layer velocity profile, we construct Table 1 for an insulated plate.

Table 1

M	$H = \delta^*/\theta$	C_F/C_{Fi} (Ref. 40)	$\delta^*/x \cong d\delta^*/dx$
0.1	1.29	1.00	0.00193
1.0	1.73	0.93	0.00242
5.0	12.31	0.37	0.00691
10.0	44.92	0.16	0.0108

We see from Table 1 that the turbulent displacement effect at $M = 10$ is five times greater than $M = 0.1$. For actual bodies, we must rely on the aerothermodynamicist to compute the proper values of δ^* considering the presence of pressure gradient, surface roughness, heat transfer, ablation, real gas thermodynamics, body geometry, and any other practical considerations. Such calculations, however, are required for any design in order to assess the heat transfer problem. In the laminar boundary (roughly $Re_x < 5 \times 10^5$ or $Re_\theta < 480$) simple estimates are often possible using the local similarity concept introduced by Lees[79] (see Ref. 80 for a recent review of this method). However, in the laminar regime, the leading edge interaction problem requires a successive approximation solution to the boundary layer growth calculation (see Ref. 70, Ch. 9, and also Ref. 81). For the more practical case of a noninsulated wall, the turbulent skin friction coefficient can be calculated roughly according to

$$C_F/C_{Fi} = (T_*/T_0)^{-0.65} \tag{172}$$

if the viscosity is proportional to $T^{0.75}$. We might take T_* as the average of the wall temperature and the maximum temperature as a reasonable estimate, although other choices have been made (see Ref. 70, pp. 328-9). From the definitions of the boundary layer displacement and momentum thickness

$$\delta^*/\delta = \int_0^1 (1 - \rho u/\rho_0 V)\, d(y/\delta) \tag{173}$$

$$\theta/\delta = \int_0^1 (1 - \rho u/\rho_0 V)(\rho u/\rho_0 V) d(y/\delta) \qquad (174)$$

utilizing the constancy of static pressure across the boundary layer and the Crocco-Busemann integral of the energy equation (Ref. 70, p. 327)

$$T/T_0 + u^2/2c_p T_0 = T_w/T_0 + (V^2/2c_p T_0)(u/V) \ . \qquad (175)$$

and again assuming the one-seventh power boundary layer profile, we can compute δ^*/θ. For $\gamma = 1.4$, $M = 20$, $T_w/T_0 = 6$ ($T_w \simeq 2400°R$) we obtain $\delta^*/\delta = 0.814$, $\theta/\delta = 0.0098$, $\delta^*/\theta = 83$, $C_F/C_{Fi} = 0.173$. We find $T_{max}/T_0 \simeq 23.6$ at $u/V = 0.467$ and $T_{max} = 9430°R = 5240°K$ (compare with $T_{stag}/T_0 = 81$, $T_w/T_0 = 6$). Then if $C_{Fi} = 0.003$ and $Re_x = 10^6$, and using the above values, we find $\delta^*/x = 0.0215$.

We now derive a relation for $d\delta^*/dx$ for the turbulent boundary layer on a flat plate in terms of δ^*/x. For a smooth plate for $5 \times 10^5 < Re_x < 5 \times 10^6$, letting $Re' = dRe_x/dx$, we have

$$C_{Fi} = 0.074/Re_x^{(0.2)} \qquad (176)$$

$$\delta^* = \frac{0.074 x^{0.8}}{(Re')^{0.2}} \left(\frac{\delta^*}{\theta} \frac{C_F}{C_{Fi}} \right) \ . \qquad (177)$$

By differentiation, we obtain

$$\frac{d\delta^*}{dx} = 0.8 \frac{0.074x^{-0.2}}{(Re')^{0.2}} \left(\frac{\delta^*}{\theta} \frac{C_F}{C_{Fi}} \right) \tag{178a}$$

$$= 0.8\delta^*/x \quad . \tag{178b}$$

Thus $d\delta^*/dx \simeq 0.016$ for the noninsulated smooth plate example. On a rough plate $C_{Fi} \simeq$ constant, and $d\delta^*/dx \simeq \delta^*/x$. We thus find a rather substantial displacement effect, equivalent to a 1.6 percent thickness ratio for the example conditions, $Re_x = 10^6$, $M = 20$, $T_w/T_0 = 6$ for a turbulent boundary layer. For these conditions, the turbulent boundary layer interaction parameter is $Md\delta^*/dx = 0.32$. Zartarian, Hsu, and Ashley[75] have noted the deterioration of piston theory for $M\theta_b > 0.3$ so that in the example shown, the viscous effect is significant in that the limits of second-order piston theory are approached.

The above calculations would be somewhat pessimistic, since dissociation of oxygen begins at $T \simeq 3500°K$ and the energy absorbed would probably limit temperatures to this vicinity. Furthermore, the vibrational degrees of freedom for oxygen and nitrogen become appreciably excited at lower temperatures (approximately 2250°K and 3370°K, respectively), and the ratio of specific heats approaches 4/3 above the higher temperature, thus somewhat reducing the temperature because of the locally higher heat capacity near T_{max}; a complete real gas calculation might show a smaller value of δ^*/θ . In this connection, Whalen[82] has shown that for air, nonequilibrium thermodynamic effects become important at speeds above 10,000 feet per second for altitudes above 150,000 feet, and that above 200,000 feet the gas composition along any streamline can be considered "frozen" at its equilibrium value on the downstream side of the shock (see also Ref. 83). Whalen shows that nonequilibrium behavior considerably affects the blunt nose influence on the steady plate flow downstream (and hence the local thermodynamic state), but does not seriously affect the leading edge viscous interaction. For simple

estimates of the combined effects of viscosity and blunting, see Bertram and Blackstock.[81]

Returning to the aeroelastic problem, we consider the dynamic pressure in relation to Re_x, M, and atmospheric properties. For $q = 1000$ psf, $M = 20$, $Re_x = 10^6$, then from Eq. (169c), we have $x = 4$ feet; this condition exists at an altitude of 145,000 feet (since $q = \gamma p_0 M^2/2$, we find $p_0 = 3.57$ psf). A recent treatment of supercircular orbital lifting re-entry vehicles[84] shows altitudes as low as 100,000 feet at $V = 30,000$ feet per second near the pullout. In such a regime, we would reach very high dynamic pressures at very high Mach numbers, and the viscous displacement would be extremely important.

In summary, we see that thickness effects are an essential feature of the hypersonic airloads problem, and these must include viscous corrections. Furthermore, any given hypersonic body is describable as having several regimes of the local hypersonic similarity parameter, $M\theta_b$, which dictates the choice of lifting pressure approximation to be used locally.

C. AICs for Slender Bodies

1. Slender-Body Theory

The equations given by Bisplinghoff, Ashley, and Halfman (Ref. 34, p. 418) provide a convenient basis for deriving the AICs of a slender body. The vertical force acting per unit length of the body is the reaction to the substantial rate of change of the momentum of the virtual mass per unit length of the body

$$\frac{dF}{dx} = -\left(V\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(\frac{dI}{dx}\right) \quad , \tag{179}$$

where the momentum of the virtual mass per unit length is found from the local cross-sectional area and the downwash

$$\frac{dI}{dx} = \rho S w \tag{180a}$$

$$= \rho S\left(V\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}\right) \quad . \tag{180b}$$

If we assume harmonic motion and substitute Eq. (180b) into Eq. (179), we obtain

$$\frac{dF}{dx} = -\rho V \frac{d}{dx}\left\{S\left(V\frac{dh}{dx} + i\omega h\right)\right\} - i\omega\rho S\left(V\frac{dh}{dx} + i\omega h\right) \quad . \tag{181}$$

To obtain the force on a specified length of the body, it is necessary to integrate Eq. (181) over that length. We consider the body to be divided into a number of sections, not necessarily of equal length, as shown in Figure 12. For the n'th section, the control point is taken at the midpoint of its length Δ_n. The aft end of the section is located at $x_{n-\frac{1}{2}} = x_n - \Delta_n/2$ and has the cross-section area $S_{n-\frac{1}{2}}$; the forward end of the section is located at $x_{n+\frac{1}{2}} = x_n + \Delta_n/2$ and has the cross-section area $S_{n+\frac{1}{2}}$. Carrying out the integration of Eq. (181) for the n'th section yields the following

$$F_n = \int_{x_{n-\frac{1}{2}}}^{x_{n+\frac{1}{2}}} \frac{dF}{dx}\, dx \tag{182a}$$

$$= -\rho V\left[S\left(V\frac{dh}{dx} + i\omega h\right)\right]_{x_{n-\frac{1}{2}}}^{x_{n+\frac{1}{2}}} - i\omega\rho \int_{x_{n-\frac{1}{2}}}^{x_{n+\frac{1}{2}}} S\left(V\frac{dh}{dx} + i\omega h\right) dx \tag{182b}$$

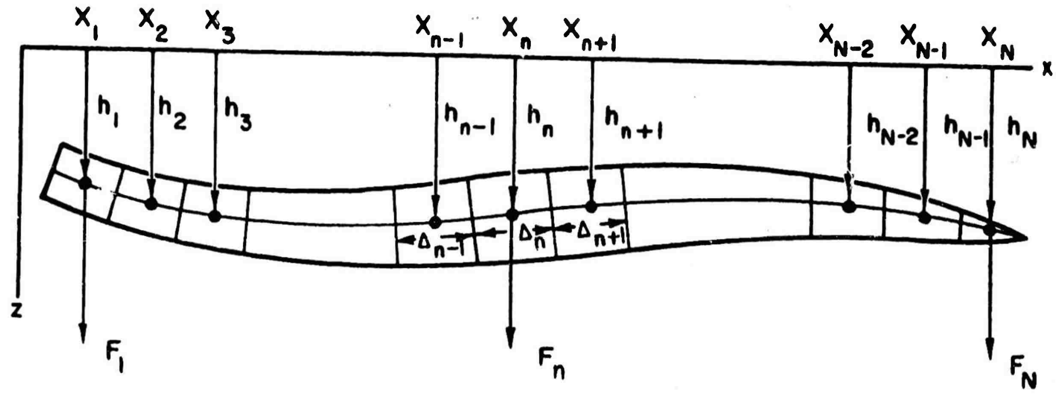

Fig. 12. Slender-Body Geometry for AICs

$$F_n = -\rho V\left[V\left(S_{n+\frac{1}{2}} h'_{n+\frac{1}{2}} - S_{n-\frac{1}{2}} h'_{n-\frac{1}{2}}\right) + i\omega\left(S_{n+\frac{1}{2}} h_{n+\frac{1}{2}} - S_{n-\frac{1}{2}} h_{n-\frac{1}{2}}\right)\right] - i\omega\rho \int_{x_{n-\frac{1}{2}}}^{x_{n+\frac{1}{2}}} S(Vh' + i\omega h)\, dx \quad . \tag{182c}$$

We resort to Lagrangian interpolation for the evaluation of the terms in Eq. (182c). For the first two terms we choose parabolic interpolation

$$h = h_{n-1} \frac{(x - x_n)(x - x_{n+1})}{(x_{n-1} - x_n)(x_{n-1} - x_{n+1})}$$

$$+ h_n \frac{(x - x_{n-1})(x - x_{n+1})}{(x_n - x_{n-1})(x_n - x_{n+1})}$$

$$+ h_{n+1} \frac{(x - x_{n-1})(x - x_n)}{(x_{n+1} - x_{n-1})(x_{n+1} - x_n)} \quad . \qquad (183)$$

For evaluation of the integral we use only linear interpolation

$$h = h_n + h_n' (x - x_n) \qquad (184)$$

and

$$S = (1/\Delta_n)\left[S_{n-\frac{1}{2}} \left(x_{n+\frac{1}{2}} - x\right) + S_{n+\frac{1}{2}} \left(x - x_{n-\frac{1}{2}}\right)\right] \quad . \qquad (185)$$

The three point interpolation leads to three elements in each row of the AIC matrix. Therefore

$$F_n = \rho\omega^2 b_r^2 s \, (C_{hn,n-1} h_{n-1} + C_{hn,n} h_n + C_{hn,n+1} h_{n+1}) \quad . \qquad (186)$$

By substituting Eq. (186) into Eq. (182c), after evaluating the integral of Eq. (182c) by means of Eqs. (184) and (185), we obtain the following relation for the AICs

$$C_{hn,n-1}\,h_{n-1} + C_{hn,n}\,h_n + C_{hn,n+1}\,h_{n+1} =$$

$$(1/k_r^2 s)\left\{ -\left(S_{n+\frac{1}{2}}\,h'_{n+\frac{1}{2}} - S_{n-\frac{1}{2}}\,h'_{n-\frac{1}{2}}\right)\right.$$

$$+ (k_r^2/b_r^2)\left[h_n\,\Delta V_n + (\Delta_n^2/12)\left(S_{n+\frac{1}{2}} - S_{n-\frac{1}{2}}\right)h'_n\right]$$

$$\left. - i(k_r/b_r)\left(S_{n+\frac{1}{2}}\,h_{n+\frac{1}{2}} - S_{n-\frac{1}{2}}\,h_{n-\frac{1}{2}} + h'_n\,\Delta V_n\right)\right\} \tag{187}$$

where ΔV_n is the volume of the n'th section.

If we evaluate the deflections in Eq. (183) at stations $x_{n-\frac{1}{2}}$ and $x_{n+\frac{1}{2}}$, and then, by differentiating Eq. (183), evaluate the slopes at stations $x_{n-\frac{1}{2}}$, x_n, $x_{n+\frac{1}{2}}$, we may then place the right-hand side of Eq. (187) in terms of the control point deflections, h_{n-1}, h_n, and h_{n+1}. If we carry out this substitution, the AICs are found by identifying the coefficients of the control point deflections on both sides of Eq. (187). The following results are obtained

$$C_{hn,\,n-1} =$$

$$(1/k_r^2 s\, D_{n-1}) \Big\{ -2S_{n+\frac{1}{2}} (\Delta_n - \Delta_{n+1}) - 2S_{n-\frac{1}{2}} (3\Delta_n + \Delta_{n+1})$$

$$- (k_r^2/b_r^2)(\Delta_n/6) \left(S_{n+\frac{1}{2}} - S_{n-\frac{1}{2}}\right) (\Delta_n + \Delta_{n+1})$$

$$+ i(k_r/b_r) \Big[S_{n+\frac{1}{2}} \Delta_n \Delta_{n+1} + S_{n-\frac{1}{2}} \Delta_n (2\Delta_n + \Delta_{n+1})$$

$$+ 2\Delta V_n (\Delta_n + \Delta_{n+1}) \Big] \Big\} \tag{188}$$

where
$$D_{n-1} = (\Delta_{n-1} + \Delta_n)(\Delta_{n-1} + 2\Delta_n + \Delta_{n+1}) \tag{189}$$

$$C_{hn,\,n} =$$

$$(1/k_r^2 s\, D_n) \Big\{ 2S_{n+\frac{1}{2}} (\Delta_{n-1} + 2\Delta_n - \Delta_{n+1}) - 2S_{n-\frac{1}{2}} (\Delta_{n-1} - 2\Delta_n - \Delta_{n+1})$$

$$+ (k_r^2/b_r^2) \left[D_n \Delta V_n + (\Delta_n^2/6) \left(S_{n+\frac{1}{2}} - S_{n-\frac{1}{2}}\right) (\Delta_{n+1} - \Delta_{n-1}) \right]$$

$$- i(k_r/b_r) \Big[S_{n+\frac{1}{2}} \Delta_{n+1} (\Delta_{n-1} + 2\Delta_n) - S_{n-\frac{1}{2}} \Delta_{n-1} (2\Delta_n + \Delta_{n+1})$$

$$+ 2\Delta V_n (\Delta_{n+1} - \Delta_{n-1}) \Big] \Big\} \tag{190}$$

where

$$D_n = (\Delta_{n-1} + \Delta_n)(\Delta_n + \Delta_{n+1}) \tag{191}$$

$$C_{hn,n+1} =$$

$$\begin{aligned}(1/k_r^2 s\, D_{n+1})\Big\{ &- 2S_{n+\frac{1}{2}} (\Delta_{n-1} + 3\Delta_n) + 2S_{\;-\frac{1}{2}} (\Delta_{n-1} - \Delta_n) \\ &+ (k_r^2/b_r^2)(\Delta_n^2/6)\left(S_{n+\frac{1}{2}} - S_{n-\frac{1}{2}}\right)(\Delta_{n-1} + \Delta_n) \\ &- i\,(k_r/b_r)\Big[S_{n+\frac{1}{2}} \Delta_n (\Delta_{n-1} + 2\Delta_n) + S_{n-\frac{1}{2}} \Delta_{n-1} \Delta_n \\ &\qquad + 2\Delta V_n (\Delta_{n-1} + \Delta_n)\Big]\Big\}\end{aligned} \tag{192}$$

where

$$D_{n+1} = (\Delta_{n-1} + 2\Delta_n + \Delta_{n+1})(\Delta_n + \Delta_{n+1}) \quad . \tag{193}$$

The above expressions are applicable for all intermediate sections of the body, sections which are centrally located as far as the interpolation and differentiation are concerned. The exceptions are the first and N'th sections. The counterpart of Eq. (187) for the first section is

$$C_{h1,1}\, h_1 + C_{h1,2}\, h_2 + C_{h1,3}\, h_3 =$$

$$(1/k_r^2 s) \left\{ - S_{\frac{3}{2}} h'_{\frac{3}{2}} + S_{\frac{1}{2}} h'_{\frac{1}{2}} + (k_r^2/b_r^2) \left[h_1 \, \Delta V_1 + (\Delta_1^2/12)(S_{\frac{3}{2}} - S_{\frac{1}{2}})\, h'_1 \right] \right.$$

$$\left. - i\,(k_r/b_r)(S_{\frac{3}{2}} h_{\frac{3}{2}} - S_{\frac{1}{2}} h_{\frac{1}{2}} + h'_1 \, \Delta V_1) \right\} \quad . \tag{194}$$

Carrying out the evaluation of the appropriate deflections and slopes from Eq. (183) in terms of the first three control point deflections leads to the following coefficients for the first row of the AIC matrix

$$C_{h1,1} =$$

$$(1/k_r^2 s\, D_1) \left\{ 2S_{\frac{3}{2}} \, (3\Delta_2 + \Delta_3) - 2S_{\frac{1}{2}} \, (4\Delta_1 + 3\Delta_2 + \Delta_3) \right.$$

$$+ (k_r^2/b_r^2) \left[D_1 \, \Delta V_1 - (\Delta_1^2/6)(S_{\frac{3}{2}} - S_{\frac{1}{2}})(2\Delta_1 + 3\Delta_2 - \Delta_3) \right]$$

$$- i\,(k_r/b_r) \left[S_{\frac{3}{2}} \, \Delta_2 \, (2\Delta_2 + \Delta_3) - S_{\frac{1}{2}} \, (2\Delta_1 + \Delta_2)(2\Delta_1 + 2\Delta_2 + \Delta_3) \right.$$

$$\left. \left. - 2\Delta V_1 \, (2\Delta_1 + 3\Delta_2 + \Delta_3) \right] \right\} \tag{195}$$

where

$$D_1 = D_{n-1} \text{ with } n = 2 \quad .$$

$$C_{h1,2} =$$

$$(1/k_r^2 s\, D_2)\Big\{2S_{\frac{3}{2}}(\Delta_1 - 2\Delta_2 - \Delta_3) + 2S_{\frac{1}{2}}(3\Delta_1 + 2\Delta_2 + \Delta_3)$$

$$+ (k_r^2/b_r^2)(\Delta_1^2/6)(S_{\frac{3}{2}} - S_{\frac{1}{2}})(\Delta_1 + 2\Delta_2 + \Delta_3)$$

$$- i\,(k_r/b_r)\Big[S_{\frac{3}{2}}\Delta_1(2\Delta_2 + \Delta_3) + S_{\frac{1}{2}}\Delta_1(2\Delta_1 + 2\Delta_2 + \Delta_3)$$

$$+ 2\Delta V_1(\Delta_1 + 2\Delta_2 + \Delta_3)\Big]\Big\} \qquad (196)$$

where

$$D_2 = D_n \text{ with } n = 2 \; .$$

$$C_{h1,3} =$$

$$(1/k_r^2 s\, D_3)\Big\{- 2S_{\frac{3}{2}}(\Delta_1 - \Delta_2) - 2S_{\frac{1}{2}}(3\Delta_1 + \Delta_2)$$

$$- (k_r^2/b_r^2)(\Delta_1^2/6)(S_{\frac{3}{2}} - S_{\frac{1}{2}})(\Delta_1 + \Delta_2)$$

$$+ i\,(k_r/b_r)\Big[S_{\frac{3}{2}}\Delta_1\Delta_2 + S_{\frac{1}{2}}\Delta_1(2\Delta_1 + \Delta_2)$$

$$+ 2\Delta V_1(\Delta_1 + \Delta_2)\Big]\Big\} \qquad (197)$$

where

$$D_3 = D_{n+1} \text{ with } n = 2 \; .$$

Similarly, the counterpart of Eq. (187) for the N'th section is

$$C_{hN,N-2}\, h_{N-2} + C_{hN,N-1}\, h_{N-1} + C_{hN,N}\, h_N =$$

$$(1/k_r^2 s)\left\{ - S_{N+\frac{1}{2}}\, h'_{N+\frac{1}{2}} + S_{N-\frac{1}{2}}\, h'_{N-\frac{1}{2}} \right.$$

$$+ (k_r^2/b_r^2)\left[h_N\, \Delta V_N + (\Delta_N^2/12)(S_{N+\frac{1}{2}} - S_{N-\frac{1}{2}})\, h'_N \right]$$

$$\left. - i(k_r/b_r)(S_{N+\frac{1}{2}}\, h_{N+\frac{1}{2}} - S_{N-\frac{1}{2}}\, h_{N-\frac{1}{2}} + h'_N\, \Delta V_N) \right\} \qquad (198)$$

and we obtain the following coefficients for the last (N'th) row of the AIC matrix

$$C_{hN,N-2} =$$

$$(1/k_r^2 s\, D_{N-2})\left\{ - 2S_{N+\frac{1}{2}}\,(3\Delta_N + \Delta_{N-1}) - 2S_{N-\frac{1}{2}}\,(\Delta_N - \Delta_{N-1}) \right.$$

$$+ (k_r^2/b_r^2)(\Delta_N^2/6)(S_{N+\frac{1}{2}} - S_{N-\frac{1}{2}})(\Delta_N + \Delta_{N-1})$$

$$- i(k_r/b_r)\left[S_{N+\frac{1}{2}}\, \Delta_N\,(2\Delta_N + \Delta_{N-1}) + S_{N-\frac{1}{2}}\, \Delta_N\, \Delta_{N-1} \right.$$

$$\left.\left. + 2\Delta V_N\,(\Delta_N + \Delta_{N-1}) \right]\right\} \qquad (199)$$

where

$$D_{N-2} = D_{n-1} \text{ with } n = N-1 \quad .$$

$$C_{hN,N-1} =$$

$$(1/k_r^2 s\, D_{N-1}) \Big\{ 2S_{N+\frac{1}{2}} (3\Delta_N + 2\Delta_{N-1} + \Delta_{N-2}) + 2S_{N-\frac{1}{2}} (\Delta_N - 2\Delta_{N-1} - \Delta_{N-2})$$

$$- (k_r^2/b_r^2)(\Delta_N^2/6)(S_{N+\frac{1}{2}} - S_{N-\frac{1}{2}})(\Delta_N + 2\Delta_{N-1} + \Delta_{N-2})$$

$$+ i(k_r/b_r)\Big[S_{N+\frac{1}{2}} \Delta_N (2\Delta_N + 2\Delta_{N-1} + \Delta_{N-2})$$

$$+ S_{N-\frac{1}{2}} \Delta_N (2\Delta_{N-1} + \Delta_{N-2})$$

$$+ 2\Delta V_N (\Delta_N + 2\Delta_{N-1} + \Delta_{N-2}) \Big] \Big\} \tag{200}$$

where

$$D_{N-1} = D_n \text{ with } n = N-1 \quad .$$

$$C_{hN,N} =$$

$$\begin{aligned}(1/k_r^2 s\, D_N)\Big\{ & -2S_{N+\frac{1}{2}}(4\Delta_N + 3\Delta_{N-1} + \Delta_{N-2}) + 2S_{N-\frac{1}{2}}(3\Delta_{N-1} + \Delta_{N-2}) \\ & + (k_r^2/b_r^2)\Big[D_N\,\Delta V_N + (\Delta_N^2/6)(S_{N+\frac{1}{2}} - S_{N-\frac{1}{2}})(2\Delta_N + 3\Delta_{N-1} + \Delta_{N-2})\Big] \\ & - i(k_r/b_r)\Big[S_{N+\frac{1}{2}}(2\Delta_N + \Delta_{N-1})(2\Delta_N + 2\Delta_{N-1} + \Delta_{N-2}) \\ & \qquad - S_{N-\frac{1}{2}}\Delta_{N-1}(2\Delta_{N-1} + \Delta_{N-2}) \\ & \qquad + 2\Delta V_N(2\Delta_N + 3\Delta_{N-1} + \Delta_{N-2})\Big]\Big\} \end{aligned} \qquad (201)$$

where

$$D_N = D_{n+1} \text{ with } n = N-1 \quad .$$

To illustrate the assembly of the coefficients into the AIC matrix, we show the format below for a slender body having five degrees of freedom

$$\left[C_h\right] = \begin{bmatrix} C_{h1,1} & C_{h1,2} & C_{h1,3} & 0 & 0 \\ C_{h2,1} & C_{h2,2} & C_{h2,3} & 0 & 0 \\ 0 & C_{h3,2} & C_{h3,3} & C_{h3,4} & 0 \\ 0 & 0 & C_{h4,3} & C_{h4,4} & C_{h4,5} \\ 0 & 0 & C_{h5,3} & C_{h5,4} & C_{h5,5} \end{bmatrix} \quad . \qquad (202)$$

Before going on to a discussion of second-order slender body theory, we note that, according to Miles,[38] the slender-body theory presented above may be applied to finned vehicles if an effective cross-section area in the finned region, whose geometry is shown in Fig. 13, is taken as

$$S_e = \pi (s^2 - R^2 + R^4/s^2) \quad . \tag{203}$$

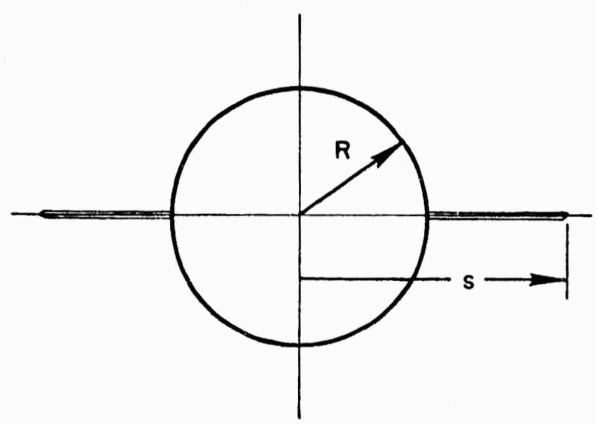

Fig. 13. Cross Section of Finned Region of Slender Body

2. Second-Order Effects of Thickness

The nonlinear effects of thickness on unsteady loads have been discussed in connection with supersonic strip theory, and a corresponding effect exists for slender bodies due to the axial flow perturbation interaction with the crossflow perturbations. These second-order theories have been pioneered by Lighthill and Van Dyke[86] for the steady state case. Van Dyke has introduced the concept of a "hybrid" theory, wherein the first-order crossflow solution is added to the second-order axial flow solution, and the exact isentropic pressure and surface boundary conditions are used. Van Dyke argues that approximations to the pressure coefficient and boundary conditions

are unnecessary even though justifiable on an order of magnitude basis. Tobak and Wehrend[72] have extended this concept to unsteady rigid motions of pitching and vertical acceleration and present plots of stability derivatives for cones including the steady state results and comparisons with exact results, and with Newtonian theory for large values of $M\theta_b$. This does not constitute a second-order crossflow theory; however, the use of the exact pressure and surface boundary condition involves the axial flow potential which is taken from the second-order theory of Van Dyke.[86] The results for a cone show a reduction in magnitude of all normal force coefficients relative to simple slender-body theory, as $M\theta_b$ increases. We now consider the second-order crossflow problem.

Revell[87] has carried out a second-order unsteady crossflow potential solution based on slender-body theory as the starting point in an iteration procedure. This solution refines both the boundary condition and the pressure coefficient expressions to obtain consistent approximation throughout. The disturbance potential Φ is split into the first-order slender-body potential ϕ, plus a second-order potential ψ. Thus, in a body axis cylindrical coordinate system (x, r, θ) inclined at an angle α the fluid motion is defined by the disturbance potential such that, following Ref. 86,

$$\vec{V}_f = V\nabla\Omega = V(\vec{1}_x \cos\alpha + \vec{1}_z \sin\alpha + \nabla\Phi) \tag{204}$$

and

$$\Phi = \phi + \psi \quad . \tag{205}$$

The potentials ϕ and ψ are resolved in terms of the axial and crossflow potentials defined by the relations

$$\phi = \phi_0 + \phi_1 \sin\theta \tag{206a}$$

$$\psi = \psi_0 + \psi_1 \sin\theta \quad . \tag{206b}$$

Now the nonlinear potential equation in fixed axes which is to be satisfied is [(See Ref. 38, p. 2, Eq. (1.1.9b)]

$$a^2 \nabla^2 \Omega = (\partial/\partial\tau + \frac{1}{2}\nabla\Omega\cdot\nabla)(\nabla\Omega)^2 + \Omega_{\tau\tau} \tag{207a}$$

where

$$a^2 = a_0^2 - (\gamma - 1)\left\{\Omega_\tau + (1/2)\left[(\nabla\Omega)^2 - 1\right]\right\} \quad . \tag{207b}$$

Substitution of Eqs. (204) to (206) into Eq. (207), expressing the time derivatives in the moving axis system ($\partial/\partial\tau = \partial/\partial t - \vec{V}_b \cdot \nabla$), and approximating the nonlinear terms with values computed from the first-order potential, ϕ, leads to the following equations

$$\Psi_{0rr} + (1/r)\Psi_{0r} = \Lambda_0(x, r) \tag{208a}$$

$$\Psi_{1rr} + (1/r)\Psi_{1r} - (1/r^2)\Psi_1 = \Lambda_1(x, r) \quad . \tag{208b}$$

The solution of Eqs. (208) has been carried out in Ref. 87, where corrections to stability derivatives have been derived for the low reduced frequency harmonic case. The results show a reduction of magnitude of stability derivatives for cones as M_{Ω_b} increases which are somewhat greater than those shown by Tobak and Wehrend,[72] partly because they have not computed the second-order crossflow potential contribution, and partly due to the rather different approaches involved. The slender-body iteration procedure is considered valid only for $M_{\Omega_b} < 0.7$, beyond which an apparent reversal of trends is indicated by the theory. The second-order slender-body solution described above is similar to the steady state second order slender-body crossflow solution developed by Lighthill;[85] therefore, the comparisons with more exact steady flow theories discussed by Van Dyke[86] are pertinent. We now consider the so-called

"first-order theory" (as opposed to "first-order slender-body theory" discussed in the preceding section).

The basic feature of the first-order problem is that it retains the wave equation character, which for the unsteady case in body axis coordinates may be derived from Eq. (4. 1. 15) of Ref. 87, and which agrees with the steady state results of Van Dyke[86] and Lighthill.[85] We have for the crossflow potential

$$\phi_{1_{rr}} + (1/r)\phi_{1_r} - (1/r^2)\phi_1 = \beta^2 \phi_{1_{xx}} + 2M^2 \phi_{1_{xt}} + M^2 \phi_{1_{tt}} \tag{209}$$

subject to the condition

$$\phi_{1_r}\Big|_{r = R} = w_b(x,t) \tag{210}$$

and the initial conditions (for <u>supersonic flow</u>)

$$\left.\begin{aligned} \phi_1(x,r,\theta,t) &= 0 \\ \phi_{1_x}(x,r,\theta,t) &= 0 \end{aligned}\right\} \quad \text{as } x \rightarrow 0 \tag{211}$$

and the evanescense at infinity

$$\lim_{r\to\infty} \phi_1(x,r,\theta,t) = 0 \tag{212}$$

The steady state solutions are known and will not be discussed (see Liepmann and Roshko,[40] for example). The general unsteady problem was first solved by Dorrance,[88] although this solution was criticized by Miles[89] on the basis that slender-body theory is the only linear theory which is consistent on the basis of order of magnitude conditions. If δ is a thickness parameter, the conditions are

$$\text{for } k = O(1)\text{: } \delta \ll 1 \text{ and } M^2\delta^2 \ll 1 \tag{213a}$$

$$\text{for } k \gg 1\text{: } k\delta \ll 1 \text{ and } k^2 M^2 \delta^2 \ll 1 \tag{213b}$$

(See also Ref. 38, Ch. 12.) Similar criticism of the steady flow analysis of Tsien[90] was given by Lighthill.[91] It is thus not clear that slender-body theory can be improved upon without going to a second-order theory. However, it does seem possible that the first-order theory, which includes the initial conditions on x and improves accuracy for high frequency cases, might provide a better starting point for iteration procedures as suggested by Revell.[87] Another unexplored possibility might be to use Dorrance's solution in conjunction with a hybrid theory as proposed by Van Dyke[86] and by Tobak and Wehrend,[72] which requires the axial flow perturbation field to at least second order.

For the purposes of aeroelastic analysis, it would be desirable to have a systematic procedure for treating flexible lifting bodies. The slender-body treatment given in Section II. C. 1 is suggested as reasonable for range where the conditions of Eqs. (213) are satisfied.* For hypersonic speeds, as has already been noted in Section II. B. 4, many other considerations enter. Recently Bond and Packard[93] have treated the first-order problem for a flexible body for which $w(x)R^2(x)$ is given by a polynomial. In their basic analysis, apparently unaware of any earlier work, they solve the identical problem treated by Dorrance.[88] They show an increase in stability derivatives for rigid motion with increasing Mach number for a parabolic nose body of revolution, as compared to first-order slender body ("apparent mass" theory in the notation of Ref. 93). This suggestion is contrary to all inferences from second-order theories, both steady and unsteady,[85-87] the hybrid theory,[72] or the limiting Newtonian values.[72] The experimental data cited in Ref. 93 are not convincing, in that there are apparent shifts in the levels of experimental values obtained in different facilities which cover the transonic as opposed to the supersonic regime. It is suggested, therefore, that for all supersonic Mach numbers,

*This can be justified based on Miles's extension of the work of Ward[92] to unsteady flow (see Ref. 38 for a bibliography).

that first-order slender-body theory probably represents an upper bound on stability on derivatives such as damping in pitch, etc., and that for high Mach numbers, each section of the body is regarded as defining some range of $M\theta_b$ for which the appropriate hypersonic theory. This is especially so in view of the importance of blunt nosed configurations. For subsonic flows, however, it is possible that the complete first-order theory might be an improvement over slender-body theory, and a numerical solution for the doublet strength distribution as with lifting surface theory would be the indicated procedure. In this connection, both Dorrance's work and that of Bond and Packard have used the slender-body estimate of the local source strength and are not complete to the extent that the doublets in the Mach forecone of each point on the body are not considered in the evaluation of the doublet strength distribution.

D. Panels

1. Two-Dimensional Panels

The two-dimensional flat panel has been considered in Ref. 94; we review this derivation of the AICs in the present notation. From the analysis of Nelson and Cunningham,[95] the complex upper surface perturbation pressure for harmonic motion can be written in dimensional form as

$$p(x) = (\rho V^2/\beta)\left\{h' + (2ik/c)\left[(M^2 - 2)/\beta^2\right]h + (4k^2/c^2)\int_0^x h\, L\left[(x - \xi)/c\right] d\xi\right\} \qquad (214)$$

where the geometry is shown in Figure 14 for the example of a single span panel, and the influence term is given by

$$L(u) = \exp(-i\bar{\omega}u)\left\{-\left[(M^2 + 2)/2\beta^4\right]J_0(\bar{\omega}u/M) + i(2M/\beta^4)J_1(\bar{\omega}u/M) + (M^2/2\beta^4)J_2(\bar{\omega}u/M)\right\} \qquad (215)$$

in which the J_n are Bessel functions of the first kind and $\bar{\omega} = 2kM^2/\beta^2$. A tabulation of this function is given in Ref. 94 for the Mach numbers 1.3, $\sqrt{2}$, 1.56, and 2.0, the reduced frequencies k = 0.1(0.1)1.0, and the argument u = -0.2(0.1)3.9.

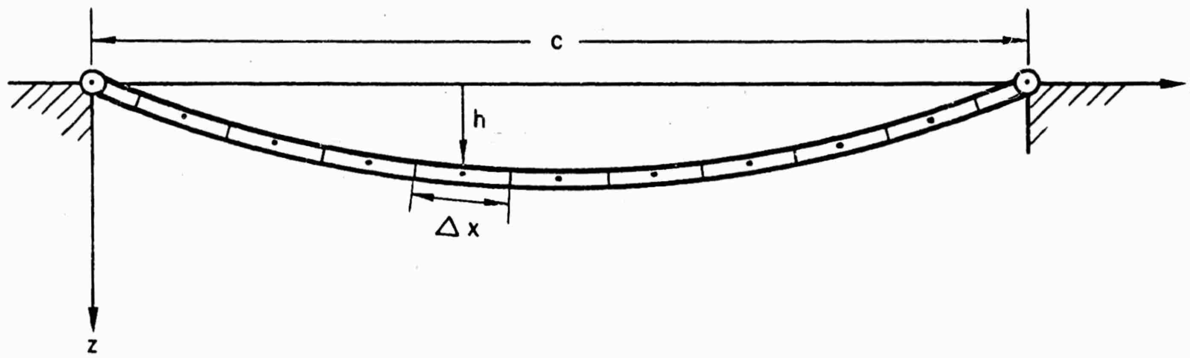

Fig. 14. Panel Geometry and Divisions for Matrix Formulation

If we neglect the effect of the air beneath the panel, assuming it to be at rest, the aerodynamic force matrix is

$$\{F\} = -\Delta x \{p\} \tag{216a}$$

$$= \rho\omega^2 (c/2)^2 \left[C_h\right] \{h\} \tag{216b}$$

from which

$$\left[C_h\right] \{h\} = -(4\Delta x/\rho\omega^2 c^2) \{p\} \tag{217}$$

Generalizing Eq. (214) to a column matrix permits substitution into Eq. (217), we obtain

$$\left[C_h\right]\left\{h\right\} = -(\Delta x/\beta k^2)\Big(\left\{h'\right\} + (2ik/c)\left[(M^2 - 2)/\beta^2\right]\left\{h\right\} + (4k^2/c^2)\int_0^x h\, L\left[(x - \xi)/c\right] d\xi\Big) \quad . \tag{218}$$

It remains to relate $\left\{h'\right\}$ and $\left\{\int h L d\xi\right\}$ to $\left\{h\right\}$, which can be accomplished by finite difference methods. It is convenient to write

$$\left[C_h\right] = \left[C_{h1}\right] + \left[C_{h2}\right] + \left[C_{h3}\right] \tag{219}$$

where

$$\left[C_{h1}\right]\left\{h\right\} = -(\Delta x/\beta k^2)\left\{h'\right\} \tag{220}$$

$$\left[C_{h2}\right]\left\{h\right\} = (2i\Delta x/ck)\left((2-M^2)/\beta^3\right)\left\{h\right\} \tag{221}$$

$$\left[C_{h3}\right]\left\{h\right\} = -(4\Delta x/\beta c^2)\left\{\int h\, L\, d\xi\right\} \quad . \tag{222}$$

Obviously

$$\left[C_{h2}\right] = (2i\Delta x/ck)\left((2-M^2)/\beta^3\right)\left[I\right] \quad . \tag{223}$$

which, for the single span example panel, after choosing $\Delta x/c = 0.1$ (i.e., the example panel has nine control points), becomes the following ninth-order constant imaginary diagonal matrix

$$\left[C_{h2}\right] = (0.2i/k)\left((2-M^2)/\beta^3\right)[I] \tag{224}$$

To investigate $\left[C_{h1}\right]$ we consider the five-point differentiation formulas of Milne[96] which are given below

$$h_1' = (1/12\Delta x)(-3h_0 - 10h_1 + 18h_2 - 6h_3 + h_4) \tag{225}$$

$$h_2' = (1/12\Delta x)(h_0 - 8h_1 + 8h_3 - h_4) \tag{226}$$

$$h_3' = (1/12\Delta x)(-h_0 + 6h_1 - 18h_2 + 10h_3 + 3h_4) \quad . \tag{227}$$

These formulas have quartic accuracy. This high order of accuracy is used because of the importance of the derivative term and the ease with which the accuracy can be utilized. It follows, then that $\left[C_{h1}\right]$ is a differentiation matrix which becomes, for the single span panel, noting $h_0 = h_{10} = 0$, the following ninth-order real matrix:

$$\left[C_{h1}\right] = (1/12\beta k^2)\begin{bmatrix} 10 & -18 & 6 & -1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & -8 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 8 & 0 & -8 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 8 & 0 & -8 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 8 & 0 & -8 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 8 & 0 & -8 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 8 & 0 & -8 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 8 & 0 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 & -6 & 18 & -10 \end{bmatrix} \tag{228}$$

To investigate $\left[C_{h3}\right]$ we consider the following integral formulas with cubic accuracy[96]

$$\int_{x_0}^{x_1} y\,dx = (\Delta x/24)(9y_0 + 19y_1 - 5y_2 + y_3) \tag{229}$$

$$\int_{x_1}^{x_n} y\,dx = (\Delta x/2)(y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$$
$$+ (\Delta x/24)(-y_0 + y_2 + y_{n-1} - y_{n+1}) \quad . \tag{230}$$

These expressions can be combined to give all of the required integrals. A typical one is

$$\int_0^{x_8} h(\xi)L\left[(x_8 - \xi)/c\right] d\xi = \Delta x \; (31/24)h_1 L(0.7) + (5/6)\,h_2 L(0.6)$$
$$+ (25/24)h_3 L(0.5) + h_4 L(0.4)$$
$$+ h_5 L(0.3) + h_6 L(0.2) + (25/24)h_7 L(0.1)$$
$$+ (1/2)h_8 L(0) - (1/24)h_9 L(-0.1) \tag{231}$$

Therefore, the integration matrix $\left[C_{h3}\right]$ for the example panel with $\Delta x/c$ = 0.1 becomes the following ninth-order complex matrix:

$$\left[C_{h3}\right] = -(0.04/\beta) \begin{bmatrix} 19L(0)/24 & -5L(-0.1)/24 & L(-0.2)/24 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4L(0.1)/3 & L(0)/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 31L(0.2)/24 & 7L(0.1)/8 & 13L(0)/24 & -L(-0.1)/24 & 0 & 0 & 0 & 0 & 0 \\ 31L(0.3)/24 & 5L(0.2)/6 & 13L(0.1)/12 & L(0)/2 & -L(-0.0)/24 & 0 & 0 & 0 & 0 \\ 31L(0.4)/24 & 5L(0.3)/6 & 25L(0.2)/24 & 25L(0.1)/24 & L(0)/2 & -L(-0.1)/24 & 0 & 0 & 0 \\ 31L(0.5)/24 & 5L(0.4)/6 & 25L(0.3)/24 & L(0.2) & 25L(0.1)/24 & L(0)/2 & -L(-0.1)/24 & 0 & 0 \\ 31L(0.6)/24 & 5L(0.5)/6 & 25L(0.4)/24 & L(0.3) & L(0.2) & 25L(0.1)/24 & L(0)/2 & -L(-0.1)/24 & 0 \\ 31L(0.7)/24 & 5L(0.6)/6 & 25L(0.5)/24 & L(0.4) & L(0.3) & L(0.2) & 25L(0.1)/24 & L(0)/2 & -L(-0.1)/24 \\ 31L(0.8)/24 & 5L(0.7)/6 & 25L(0.6)/24 & L(0.5) & L(0.4) & L(0.3) & L(0.2) & 25L(0.1)/24 & L(0)/2 \end{bmatrix} \qquad (232)$$

The complete ninth-order AIC matrix for the single span panel follows by summing Eqs. (224), (228), and (232) according to Eq. (219). The extension of the foregoing to multiple span two-dimensional panels is straightforward. The AICs for a two span panel having 18 control points are also shown in Ref. 94.

2. Three-Dimensional Panels

The AICs for a three-dimensional panel have not been obtained specifically. The two-dimensional analysis provided the simplification of a non-singular kernel in the integral term; the three-dimensional analysis requires the more complicated approaches of supersonic lifting surface theory discussed previously in Section II. B. 3. In principle, these methods are directly applicable, but there are practical numerical difficulties because a large number of downwash control points are needed to provide an adequate basis for numerical solution for the pressure loading, in comparison with the requirements for analysis of wing modes, for example. This follows from the fact that we must have a reasonable number, say six, control points per half wave of deformation (cf., Ref. 56 in connection with the box method). For example, if we consider the isolated panel of Fig. 15 to deform in modes of the form (e. g., for simply supported edges)

$$h(x, y) = \sum_{r=1}^{N_r} \sum_{s=1}^{N_s} d_{rs} \sin(r\pi x/a)\sin(s\pi y/b) \qquad , \qquad (233)$$

then we require 36 $N_r N_s$ downwash control points to obtain adequate results. If it is necessary to consider the interference of N_p panels, then 36 $N_p N_r N_s$ becomes the required total number of control points. By virtue of this large number of control points, we thus conclude that the lifting surface theories treated in Section II. B. 3 are feasible though computationally more time consuming (and expensive) than most lifting surface studies. This application of lifting surface theory to panel flutter investigations, which does not appear to have been made thus far, deserves further research. The development of the AICs follows the development of Section II. B. 3, once the matrix $\left[D^i_{nm}\right]$ has

been computed, although we note that we must consider the upper- and lower-surface panels separately, and thus have only half of the loading normally developed for a given downwash distribution assuming that the effect of the internal air may be neglected.

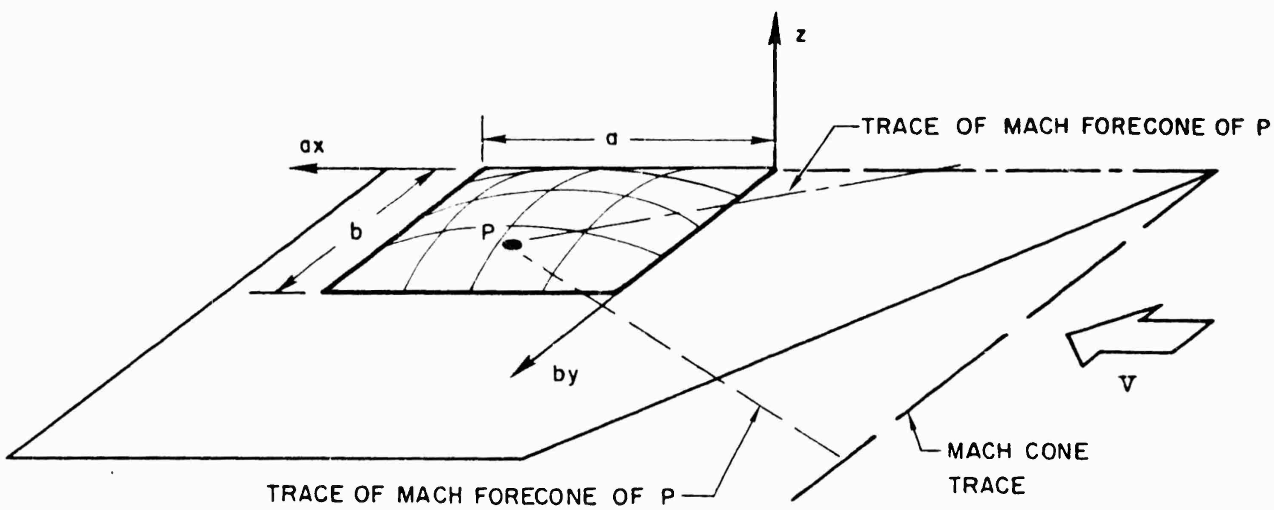

Fig. 15. Three-Dimensional Panel Geometry

We shall not consider the cylindrical panel aerodynamic problem in detail except to note as an example that the supersonic problem has been formulated, but not completely solved, by Holt and Strack.[97] Their solution has been carried out by Laplace transform methods (transforming the chordwise coordinate) for quasisteady motion. Their treatment of the boundary condition at the cylindrical surface is actually more rigorous than the treatments mentioned in Section II. C. 2 for the slender-body problem for the low frequency case, but because they choose to neglect the potential time derivative term in the calculation of the pressure perturbation, the analysis is not an altogether consistent low frequency theory. This first-order analysis should be extended to arbitrary frequencies, and the estimation of second-order effects must be considered. No attempt as yet has been made to develop the AICs for cylinders.

3. Viscous Effects

If the longitudinal and transverse wave lengths of panel modes become comparable in magnitude to the local boundary layer thickness, then viscous effects must be considered. We have already noted Miles'[47] work in this area in connection with the discussion of vorticity effects in Section II. B. 2. c. The boundary layer acts as a nonuniform free stream upon which small inviscid disturbances are superimposed. One of the features of this class of problems is that the pressure gradient normal to the surface is not negligible as is the case in classical thin boundary layer situations.[77] This assumption is the cornerstone of the justification for calculating pressure distributions from inviscid aerodynamic theory. The extension of Miles' approach to consider the viscous disturbances due to surface motion for laminar boundary layers can follow along the lines of classical hydrodynamic stability theory.[98] Once this extension is made, the AICs will follow as before.

III. OSCILLATORY AICs FOR GUSTS

Recent advances in the mathematics of random processes have provided the basis for statistical solutions to engineering problems heretofore analyzed by rather arbitrary deterministic methods. Examples of interest to the aeroelastician include gust stresses, fatigue life, and servomechanism performance in a turbulent atmospheric environment. For the analysis of these problems it is necessary to know both the statistical properties of the forcing function (i.e., power spectral density of the turbulence) and the frequency response characteristics of the system transfer function (e.g., the stress at a point due to a harmonic gust with a given wave length). The frequency response analysis for a restrained system has been briefly indicated in Section I. Methods of fatigue life analysis or servomechanism noise analysis are beyond the scope of the present treatment, but having illustrated the need for harmonic gust analysis, we may proceed to consider the derivation of oscillatory AICs for a harmonic gust.

We consider first the case of strip theory. We take the derivation from Ref. 3. In Section I, the gust AICs were defined by

$$\left\{F_g\right\} = \rho V W_g b_r s \left[C_g\right] \left\{w_g / W_g\right\}$$

where w_g is the gust downwash velocity at each spanwise station; the column $\left\{w_g / W_g\right\}$ becomes a unit column matrix for a uniform (i.e., two-dimensional) gust field. We have noted this definition is completely general, being equally as applicable to a lifting surface theory as to a strip theory. However, in the case of a strip theory, the AICs take on the simplified partitioned form. For example, the gust AICs for a two strip wing appear as

$$\left[C_g\right] = \left[\begin{array}{c|c} C_{g1} & O \\ \hline O & C_{g2} \end{array}\right] \tag{234}$$

Each partition is of size (2 x 1) since two control points are necessary on each strip if it has two flexible degrees of freedom (assuming a rigid chord, and no control surface).

In order to derive the strip AICs we assume that the gust lift and moment about the airfoil quarter chord are known. The equivalence between the given loads and the control point forces is shown in Fig. 16. We note the sign convention is chosen arbitrarily, only consistency being required; if the gust is

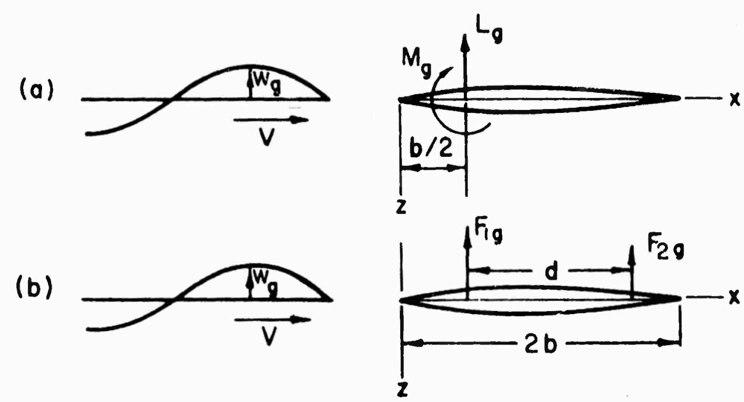

Fig. 16. Given (a) and Replacement (b) Force Systems and Geometry

specified as an upwash, a positive force acts upward, and if the gust is specified as a downwash, a positive force acts downward. From Fig. 16 we note the equivalence

$$F_{1g} + F_{2g} = L_g \tag{235}$$

$$dF_{2g} = -M_g \tag{236}$$

which, in matrix form, reads

$$\begin{Bmatrix} F_{1g} \\ F_{2g} \end{Bmatrix} = \begin{Bmatrix} L_g + M_g/d \\ -M_g/d \end{Bmatrix} \quad . \tag{237}$$

In order to continue the derivation it is necessary to have the relationships between the lift and moment and the gust velocity. To this end we select the incompressible solution of Sears[99] who gives the lift and moment (when corrected for sweep) as

$$L_g = 2\pi \cos\Lambda \rho V w_g b \Delta y \Phi(k) \exp(-ikx/b) \tag{238}$$

$$M_g = 0 \tag{239}$$

where the Sears function is given in terms of the Theodorsen function C(k) as

$$\Phi(k) = \left[J_0(k) - iJ_1(k)\right] C(k) + iJ_1(k) \quad . \tag{240}$$

The first sweep correction, the factor $\cos\Lambda$, adjusts the two-dimensional lift curve slope; the second sweep correction, the factor exp(-ikx/b), accounts for the phase difference in the gust maximum amplitude reaching the leading edge of each wing strip. Substituting Eqs. (238) and (239) into Eq. (237) yields the force-gust velocity relationship

$$\begin{Bmatrix} F_{1g} \\ F_{2g} \end{Bmatrix} = 2\pi \cos\Lambda \rho V w_g \begin{Bmatrix} b\Delta y \Phi(k) \exp(-ikx/b) \\ 0 \end{Bmatrix} \tag{241}$$

which, by comparison to Eq. (3), yields the harmonic gust AICs for the j'th strip

$$\left\{C_{gj}\right\} = 2\pi \cos\Lambda \begin{Bmatrix} (b_j/b_r)(\Delta y_j/s)\ \Phi(k_j) \exp(-ik_j x_j/bj) \\ 0 \end{Bmatrix} , \tag{242}$$

where the local reduced frequency, k_j, is based on the local semichord length

$$k_j = \omega b_j/V \tag{243a}$$

$$= (2\pi V/\lambda)(b_j/V) \tag{243b}$$

$$= 2\pi b_j/\lambda \quad , \tag{243c}$$

where λ is the gust wave length. As in the case of motion, once the AIC partitions are obtained for each strip, the surface AICs are formed by assembling the partitions according to Eq. (234).

We may consider the formulation from lifting surface theory more briefly, since it is only necessary to replace the downwash from motion in Eq. (55) with the gust downwash matrix $\left\{ w_g \exp(-ik_j x_j / b_j) \right\}$ where the x_j are now the chordwise coordinates of the control points. Then the harmonic gust AIC matrix from lifting surface theory becomes

$$\left[C_g \right] = K \left[B^j_{nm} \right] \left[A^i_{nm} \right] \left[\exp(-ik_j x_j / b_j) \right] \tag{244}$$

where the matrices $\left[B^j_{nm} \right]$ and $\left[A^i_{nm} \right]$, and the scale factor K are defined in Section II.B.1 for subsonic flow and in Section II.B.3 for supersonic flow, and the exponential matrix is a diagonal form. A similar treatment is possible in the case of slender body theory by setting $w = W_g \exp(-k_r x / b_r)$ in Eq. (180) of Section II.C and proceeding according to the method of that section.

IV. TRANSIENT AICs

The current status of transient AICs permits only a few preliminary remarks. For the moment, we shall discuss some promising means available to utilize our oscillatory solutions in deriving useful transient results, rather than seeking transient solutions per se.

For use in dynamic stability and control investigations, a Taylor series format may be obtained for the transient AICs for motion. The series may be written

$$\left\{F\right\} = (qS/\bar{c}) \left(\left[C_{hs}\right] \left\{h\right\} + \left[C_{hDh}\right]\left\{\dot{h}\bar{c}/V\right\} \right.$$

$$\left. + \left[C_{hD^2h}\right]\left\{\ddot{h}\bar{c}^2/V^2\right\} + \left[C_{hD^3h}\right]\left\{\dddot{h}\bar{c}^3/V^3\right\} + \cdots \right) \qquad (245)$$

where all matrices thus defined are real. In the case of harmonic motion, Eq. (245) must agree with Eq. (2), at least in the frequency range of interest, as we have already shown in the limit of zero frequency [i. e., Eq. (5)]. Hence

$$\left[C_{hs}\right] + ik_r(\bar{c}/b_r) \left[C_{hDh}\right] - k_r^2\left(\bar{c}/b_r\right)^2 \left[C_{hD^2h}\right]$$

$$-ik_r^3\left(\bar{c}/b_r\right)^3 \left[C_{hD^3h}\right] + \cdots = 2k_r^2(\bar{c}s/S) \left[C_h(k_r)\right] \qquad (246)$$

where we emphasize the functional dependence of $\left[C_h\right]$ on the reduced frequency. The equality of matrices in Eq. (246) requires the equality of each element

$$\left(C_{hs}\right)_{ij} + ik_r\left(\bar{c}/b_r\right) \left(C_{hDh}\right)_{ij} - k_r^2\left(\bar{c}/b_r\right)^2\left(C_{hD^2h}\right)_{ij}$$

$$-ik_r^3\left(\bar{c}/b_r\right)^3\left(C_{hD^3h}\right)_{ij} + \cdots = 2k_r^2(\bar{c}s/S) \left(C_h(k_r)\right)_{ij}. \qquad (247)$$

Eq. (247) is the basis of solving for the unknown matrices of Eq. (245). Depending on the point of truncation of the series on the left-hand side of Eq. (247), a sufficient number of reduced frequencies may be chosen to yield a determinate system of simultaneous equations for the unknown elements. For example, choosing only one frequency leads to C_{hDh} and C_{hD^2h} by equating the real and imaginary parts of both sides of Eq. (247) (noting that C_h is complex, and assuming C_{hs} has been determined previously); choosing two frequencies leads to two simultaneous equations in C_{hDh} and C_{hD^3h}, and two more simultaneous equations in C_{hD^2h} and C_{hD^4h}, from which all four coefficients may be calculated. We anticipate transient AICs derived in this manner to have practical value in the study of the dynamic stability and control of slowly maneuvering vehicles (i. e., the maneuver may be characterized by a low reduced frequency).

A more general approach which is not subject to the frequency restriction of the above method is based on the convolution formulation. Here, we may define transient AICs by the following expression [cf., Ref. 34, Eq. (5-370)]

$$\{F\} = (qS/\bar{c}) \left[C_h\left(V(t-t_0)/b_r\right)\right] \{h(t_0)\} + (qS/\bar{c}) \int_{t_0}^{t} \left[C_h\left(V(t-\tau)/b_r\right)\right] \left\{\frac{dh(\tau)}{d\tau}\right\} d\tau \quad . \tag{248}$$

If we assume the following exponential approximation for each element of the transient AIC matrix

$$C_h(Vt/b_r)_{ij} = (C_{hs})_{ij} \left[1 - \sum_k b_{ijk} \exp(-\beta_{ijk} Vt/b_r)\right] \quad , \tag{249}$$

we are in a position to align the harmonic case of Eq. (248) with the oscillatory case of Eq. (2). The form chosen in Eq. (249) leads to asymptotic convergence to the static case. In order to evaluate Eq. (248) for harmonic motion, we let

$t_0 \rightarrow -\infty$, and set $\left\{h(t)\right\} = \mathrm{Re}\left\{\bar{h} \exp(i\omega t)\right\}$. Noting also that $\left\{F(t)\right\} = \mathrm{Re}\left\{\overline{F} \exp(i\omega t)\right\}$, we find the integration leads to the following result for the complex force amplitude:

$$\left\{\bar{F}\right\} = (qS/\bar{c})\left[C_{hs}\left(1 - \sum_k b_k/(1 - i\beta_k/k_r)\right)\right]\left\{\bar{h}\right\} \quad . \tag{250}$$

Again the conformity between Eq. (250) and Eq. (2) requires equality of each element. Hence

$$1 - \sum_k \frac{b_{ijk}}{1 - i\beta_{ijk}/k_r} = 2k_r^2(\bar{c}s/S)\frac{C_h(k_r)_{ij}}{(C_{hs})_{ij}} \quad . \tag{251}$$

As before, the point of truncation of the series determines the number of reduced frequencies necessary to solve for the coefficients of the exponential series; each additional term (having two coefficients) requires one additional frequency [noting again Eq. (251) is complex]. For example, a two-term series (perhaps a practical maximum) requires two frequencies to solve for the four coefficients in each element, b_{ij1}, β_{ij1}, b_{ij2}, and β_{ij2}. The reduced frequencies may be chosen to cover a wide range without restriction, although good low frequency behavior is still desirable. A similar development to the present one is possible to obtain transient gust AICs from the oscillatory gust AICs. Current investigations have not progressed to this point, nor have they progressed sufficiently to assess practical difficulties in applying Eq. (251) (e.g., what are the consequences of an element $(C_{hs})_{ij}$ vanishing?). Investigations of transient AICs *per se* are certainly not to be discouraged, but perhaps should await demonstration of the utility of the foregoing possibilities.

The problem of obtaining the transient response of a system should also be discussed, since the responses to discrete gusts and/or blast waves are problems of continuing concern to the aeroelastician. We mention here an alternative to the integration of the equations of motion to obtain the time history. We recognize Eq. (17) as a Fourier transform of the transient response problem.

If the solution for the response matrix $\left\{ h_f(\tau) \right\}$ is carried out for several values of frequency, the transient response $\left\{ (h_f(t) \right\}$ may be obtained by means of the numerical inverse Fourier transform techniques of Lanczos (Ref. 37, p. 263). The calculation procedure is given by

$$\left\{ h_f(t) \right\} = (1/2T) \sum_{k_r = -k_{max}}^{k_{max}} \left\{ h_f(k/2T) \right\} \exp (\pi ikt/T) \qquad (252)$$

where T is a representative decay time, selected so that $\left\{ h_f(t) \right\} \simeq 0$ when $t < -T$ and $t > +T$, and $k_{max}/2T$ is taken as three times the largest participating natural frequency of the system. The investigations of L. G. Johnson and T. E. Stenton of North American Aviation, Inc., (private communication) have shown that it is not necessary to evaluate the AICs at each of the frequencies required in Eq. (252) since the AICs are smoothly varying functions of frequency and may be readily interpolated by means of any of several available numerical interpolation methods. Transient AICs may also be obtained directly by this method for use in Eq. (248) since the Fourier transform technique is an alternative to the exponential approximation of Eqs. (249) to (251).

V. AN EMPIRICAL WEIGHTING MATRIX FOR USE WITH THEORETICAL AICs

Historically speaking, both aerodynamic and structural engineers have utilized wind tunnel data in devising semi-empirical methods of aeroelastic analysis for use in dynamic stability and load studies, whereas the flutter engineer has been forced to rely on theoretical aerodynamic data. The variation in emphasis between experimental and theoretical approaches is justified to some extent because of the range of reduced frequencies appropriate to each problem. However, the semi-empirical quasi-steady methods have the distinct disadvantage of being attempts at deriving the general from the particular, but, correspondingly, the theoretical methods have the disadvantage of not approaching the experimental values as the reduced frequency is made smaller. It is suggested that a unification is possible if the theoretical data are adjusted to match the available experimental results, and thus the generality of the original formulation may be retained.

Following Refs. 100 and 101, we propose that a premultiplying diagonal weighting matrix be introduced into Eqs. (1) and (2) to adjust the theoretical AICs to yield agreement with experimental data. The corrected static force distribution becomes

$$\left\{F\right\} = (qS/\bar{c}) \left[W\right] \left[C_{hs}\right] \left\{h\right\} \quad . \tag{253}$$

and the corrected oscillatory force distribution for motion becomes

$$\left\{F\right\} = \rho\omega^2 \, b_r^2 \, s \left[W\right] \left[C_h\right] \left\{h\right\} \quad . \tag{254}$$

Eqs. (253) and (254) indicate the form in which the weighting matrix is to be utilized in aeroelastic analysis. If the experimental data correspond to a rigid untwisted configuration, we set $\left\{\bar{h}\right\} = \alpha \left\{x\right\}$ in Eq. (253) and pose the problem as the solution of

$$\left\{F\right\} = qS\alpha \left[W\right] \left[C_{hs}\right] \left\{x/\bar{c}\right\} \tag{255}$$

for $\left[W\right]$ given all the remaining terms in the equation. We consider two possibilities regarding the extent of our experimental knowledge:

1. Only a meager amount of data are available, e.g., the lift curve slope, and the spanwise and chordwise location of the aerodynamic center, or

2. A wealth of data are available, e.g., the spanwise variations of lift distribution and chordwise location of local aerodynamic center. Case 2 presents no difficulty since a sufficient number of equations can be written from Eq. (255) to determine all elements of $\lceil W \rfloor$. (Note: In this case, the weighting matrix need not be diagonal, and, depending on the aerodynamic theory used, other formats may be preferable.) In Case 1 there are considerably more unknowns to be found than equations, and, hence, we again turn to the method of least squares, although we note we are here dealing with an underdetermined system in contrast to the overdetermined systems discussed previously. We propose our additional equations from the requirement that the changes in the theoretical load distribution shall be as uniform as possible or, in least squares terminology, the weighted sum of the squares of the deviations shall be a minimum, where the deviation is defined as the difference of final and original weighting factors

$$\{\epsilon\} = \{W - I\} \quad . \tag{256}$$

The weighting function we choose is the theoretical load distribution. (This choice leads to equal weighting elements for the case where only the lift curve slope is known.) The weighted least squares condition then becomes

$$\sum F^{(t)} \epsilon^2 = \{\epsilon\}^* \left[F^{(t)}\right] \{\epsilon\} = \text{a minimum} \tag{257}$$

where the middle matrix factor is a diagonal form of

$$\left\{F^{(t)}\right\} = qS\alpha \left[C_{hs}\right] \{x/\bar{c}\} \quad . \tag{258}$$

It is now convenient to make a distinction between the weighting factors associated with the number of given conditions and those associated with the remaining least squares conditions. In analogy with the problem of redundant structures we refer to the given conditions as basic equations and the corresponding weighting factors as basic weighting factors W_b, and the additional equations as redundant equations with W_r as the corresponding redundant weighting factors. Introducing this distinction into Eqs. (256) and (257) we have

$$\left\{\epsilon_b\right\} = \left\{W_b - I\right\} \tag{259a}$$

$$\left\{\epsilon_r\right\} = \left\{W_r - I\right\} \tag{259b}$$

and

$$\left\{\epsilon_b\right\}^* \left[F_b^{(t)}\right] \left\{\epsilon_b\right\} + \left\{\epsilon_r\right\}^* \left[F_r^{(t)}\right] \left\{\epsilon_r\right\} = \text{a minimum} \quad . \tag{260}$$

If we differentiate Eq. (260) with respect to each of the redundant weighting factors, we obtain the redundant equations

$$\left[\frac{\partial \epsilon_b}{\partial W_r}\right]^* \left[F_b^{(t)}\right] \left\{\epsilon_b\right\} + \left[\frac{\partial \epsilon_r}{\partial W_r}\right]^* \left[F_r^{(t)}\right] \left\{\epsilon_r\right\} = 0 \quad . \tag{261}$$

where, from Eqs. (259)

$$\left[\frac{\partial \epsilon_b}{\partial W_r}\right] = \left[\frac{\partial W_b}{\partial W_r}\right] \tag{262a}$$

$$\left[\frac{\partial \epsilon_r}{\partial W_r}\right] = \left[I\right] \tag{262b}$$

since the W_b and $\overline{W}_r$ are considered the dependent and independent variables, respectively.

We next consider the basic equations. We may write these in the form

$$\left[Q^{(t)}\right] \left\{W\right\} = \left\{Q^{(e)}\right\} \tag{263}$$

or by partitioning

$$\left[Q_b^{(t)}\right] \left\{W_b\right\} + \left[Q_r^{(t)}\right] \left\{W_r\right\} = \left\{Q^{(e)}\right\} \tag{264}$$

where Q denotes a generalized force (lift, pitching moment, rolling moment, etc.) and $\left\{Q^{(e)}\right\}$ is the given set of experimental conditions to be matched. For example, if the three conditions given are the lift curve slope and aerodynamic center coordinates, then

$$\left\{Q^{(e)}\right\} = C_{Z_\alpha} \alpha q S \begin{Bmatrix} 1 \\ \bar{x} \\ \bar{y} \end{Bmatrix} \quad . \tag{265}$$

The simultaneous solution of Eqs. (261) and (264) is straightforward. From Eq. (264)

$$\left\{ W_b \right\} = \left[Q_b^{(t)} \right]^{-1} \left(\left\{ Q^{(e)} \right\} - \left[Q_r^{(t)} \right] \left\{ W_r \right\} \right) \tag{266}$$

from which, by differentiation,

$$\left[\frac{\partial W_b}{\partial W_r} \right] = - \left[Q_b^{(t)} \right]^{-1} \left[Q_r^{(t)} \right] \quad . \tag{267}$$

Then, combining Eqs. (259), (262), and (266) into Eq. (261) permits solution for the redundant weighting factors

$$\left\{ W_r \right\} = \left[A \right]^{-1} \left\{ B \right\} \quad , \tag{268}$$

where

$$\left[A \right] = \left[F_r^{(t)} \right] + \left[\frac{\partial W_b}{\partial W_r} \right]^* \left[F_b^{(t)} \right] \left[\frac{\partial W_b}{\partial W_r} \right] \tag{269a}$$

and

$$\begin{aligned} \left\{ B \right\} = {} & \left[\frac{\partial W_b}{\partial W_r} \right]^* \left\{ F_b^{(t)} \right\} + \left\{ F_r^{(t)} \right\} \\ & - \left[\frac{\partial W_b}{\partial W_r} \right]^* \left[F_b^{(t)} \right] \left[Q_b^{(t)} \right]^{-1} \left\{ Q^{(e)} \right\} \end{aligned} \tag{269b}$$

and where the derivative matrix is given in Eq. (267). The basic weighting factors are found by substituting Eq. (268) back into Eq. (266), and the weighting matrix is finally found by properly assembling the basic and redundant weighting factors into the diagonal format.

It has been shown that a weighting matrix can be defined which provides a means of unifying the use of theoretical AICs and experimental data. The result of this unification is that the resulting empirically modified matrices of static or oscillatory AICs retain the generality of the original theoretical basis while including the experimental results as limiting values.

VI. CONCLUDING REMARKS

We conclude that the status of AICs is reasonably well advanced, particularly in the case of oscillatory motion of lifting surfaces. A multiplicity of theoretical solutions for AICs has been established, and a measure of unification has been achieved for a diversity of configurations and throughout the range of Mach number of current interest; the empirical modification of the theoretical AICs suggested further extends the unification. The generality of AICs makes it highly desirable that subsequent investigations be carried through to this point, particularly for computational efficiency. It is hoped that the format presented here can provide a canonical form for continuing developments.

A number of recommendations can be made for further study beyond those already suggested in the text. These range from research on fundamental problems to numerical implementation of established solutions in the AIC format. Of particular importance are developments in the transonic and hypersonic regimes. In the transonic connection, the work of Landahl[39] has provided a substantial basis for improved transonic calculations. In the hypersonic connection, the aeroelastician must take his lead from the field of aerothermodynamics for further developments. In addition, there is an urgent need for further study of slender bodies and the interference of bodies and low-aspect ratio fins. It is also recommended that a number of harmonic gust AIC solutions be established based on the lifting theories discussed here. Finally, the possibilities suggested for transient aerodynamic calculations warrant further investigation.

It is difficult to assess the need for experimental correlation. Most of the theories discussed here have already obtained some measure of correlation, and this correlation also applies when the theory is cast into the AIC format. However, one candidate for a correlation study is the empirical weighting matrix discussed in Section V. The correlation in the static case will have to be carried out from force measurements on twisted models; the correlation in the oscillatory case may be made from flutter data. Such studies should show not only the accuracy of this method but also may indicate means of finding a more general format for the weighting matrix than the real diagonal form presently considered.

REFERENCES

1. W.P. Rodden. "Aerodynamic Influence Coefficients from Strip Theory." Journal of the Aerospace Sciences, 26 (1959), 833.

2. W.P. Rodden. "A Matrix Approach to Flutter Analysis." Institute of the Aerospace Sciences Fairchild Fund Paper No. FF-23, May 1958; based on North American Aviation, Inc., Report NA-56-1070, 1 May 1956.

3. W.P. Rodden, E.F. Farkas, and F.C. Slack. "Harmonic Gust Aerodynamic Influence Coefficients by Incompressible Strip Theory: Analytical Development and Procedure for the IBM 7090 Computer." Northrop Corporation, Report NOR-61-59, 14 April 1961.

4. V.M. Falkner. "The Calculation of Aerodynamic Loading on Surfaces of Any Shape." British ARC, R & M No. 1910, 1943.

5. J. Weissinger. "The Lift Distribution of Swept-Back Wings." NACA TM 1120, 1947.

6. H. Multhopp. "Methods for Calculating the Lift Distribution of Wings (Subsonic Lifting Surface Theory)." British ARC, R & M No. 2884, January 1950.

7. F.W. Diederich and M. Zlotnick. "Calculated Spanwise Lift Distributions and Aerodynamic Influence Coefficients for Unswept Wings in Subsonic Flow." NACA TN 3014, September 1953.

8. L.J. Kulakowski and R.N. Haskell. "Solution of Subsonic Nonplanar Lifting Surface Problems by Means of High-Speed Digital Computers." Journal of the Aerospace Sciences, 28 (1961), 103.

9. J.C. Evvard. "Use of Source Distributions for Evaluating Theoretical Aerodynamics of Thin Finite Wings at Supersonic Speeds." NACA TR 951, 1950.

10. B. Etkin and F.A. Woodward. "Lift Distribution on Supersonic Wings with Subsonic Leading Edges and Arbitrary Angle of Attack Distribution." Journal of the Aeronautical Sciences, 21 (1954), 783.

11. R.F. Farrell, M.B. Zisfein, and V.M. Donato. "A Rapid Method for Determination of Supersonic Line Aerodynamic Influence Coefficients." WADC TR 58-95, December 1958.

12. E.E. Groth. "Aerodynamic Influence Coefficients for Wings with Supersonic Leading and Trailing Edges and Streamwise Wing Tips." Northrop Corporation, Report ASDR-59-6, 28 August 1959.

13. T. Theodorsen. "General Theory of Aerodynamic Instability and the Mechanism of Flutter." NACA Report 496, 1934.

14. T. Theodorsen and I.E. Garrick. "Mechanism of Flutter -- A Theoretical and Experimental Investigation of the Flutter Problem." NACA Report 685, 1938.

15. T. Theodorsen and I.E. Garrick. "Nonstationary Flow about a Wing-Aileron-Tab Combination Including Aerodynamic Balance." NACA Report 736, 1941.

16. B. Smilg and L.S. Wasserman. "Application of Three-Dimensional Flutter Theory to Aircraft Structures," U.S. Army Air Force TR 4798, 8 July 1942.

17. I.N. Spielberg. "The Two-Dimensional Incompressible Aerodynamic Coefficients for Oscillatory Changes in Airfoil Camber." Journal of the Aeronautical Sciences, 20 (1953), 432.

18. R. Timman, A.I. van de Vooren, and J.H. Greidanus. "Aerodynamic Coefficients of an Oscillating Airfoil in Two-Dimensional Subsonic Flow." Journal of the Aeronautical Sciences, 21 (1954), 499.

19. N. Rott. "Oscillating Airfoils at Mach Number One." Journal of the Aeronautical Sciences, 16 (1949), 380.

20. H.C. Nelson and J.H. Berman. "Calculations of the Forces and Moments for an Oscillating Wing-Aileron Combination in Two-Dimensional Potential Flow at Sonic Speed." NACA TN 2590, 1952.

21. I.E. Garrick and S.I. Rubinow. "Flutter and Oscillating Air-Force Calculations for an Airfoil in Two-Dimensional Supersonic Flow." NACA Report 846, 1946.

22. V. Huckel and B.J. Durling. "Tables for Wing-Aileron Coefficients of Oscillating Air Forces for Two-Dimensional Supersonic Flow." NACA TN 2055, 1950.

23. M.D. Van Dyke. "Supersonic Flow Past Oscillating Airfoil Including Nonlinear Thickness Effects." NACA TN 2982, 1953.

24. W.P. Rodden and J.D. Revell. "Oscillatory Aerodynamic Coefficients for a Unified Supersonic Hypersonic Strip Theory." Journal of the Aerospace Sciences, 27 (1960), 451; based on North American Aviation, Inc., Report NA-57-1549, 31 December 1957.

25. H. Ashley and G. Zartarian. "Piston Theory -- A New Aerodynamic Tool for the Aeroelastician." Journal of the Aeronautical Sciences, 23 (1956), 1109.

26. H.G. Kuessner. "Allgemeine Tragflachentheorie." Luftfahrtforschung, 17 (1940), 37038.

27. C.E. Watkins, H.L. Runyan, and D.S. Woolston. "On the Kernel Function of the Integral Equation Relating the Lift and Downwash Distributions of Oscillating Finite Wings in Subsonic Flow." NACA Report 1234, 1955.

28. H.L. Runyan and D.S. Woolston. "Method for Calculating the Aerodynamic Loading on an Oscillating Finite Wing in Subsonic and Sonic Flow." NACA Report 1322, 1957.

29. W.P. Rodden, C.H. Hodson, and J.D. Revell. "Subsonic and Sonic Aerodynamic Influence Coefficients from Unsteady Lifting Surface Theory." North American Aviation, Inc., Report NA-57-1104, 25 October 1957.

30. C.E. Watkins, D.S. Woolston, and H.J. Cunningham. "A Systematic Kernel Function Procedure for Determining Aerodynamic Forces on Oscillating or Steady Finite Wings at Subsonic Speeds." NASA TR R-48, 1959.

31. J.D. Revell, R.J. Carrington, D.S. Milligan, and T. Vivian. "A Digital Calculation of Three-Dimensional, Subsonic, Unsteady Aerodynamic Influence Coefficients by an Extension of the NASA Kernel Function Procedure." Norair Division, Northrop Corporation, Report NOR-61-104, 14 December 1961.

32. P.T. Hsu. "Flutter of Low-Aspect Ratio Wings -- Part I: Calculations of Pressure Distributions for Oscillating Wings of Arbitrary Planform in Subsonic Flow by the Kernel Function Method." M.I.T. Aeroelastic and Structures Research Laboratory, Report 64-1, October 1957; also, "Pressure Distribution and Flutter Analysis of Low-Aspect Ratio Wings." M.I.T. Aeroelastic and Structures Research Laboratory, Report 64-3, June 1959.

33. P.T. Hsu. "Some Recent Developments in Flutter Analysis of Low-Aspect Ratio Wings." Proceedings of the National Specialists Meeting on Dynamics and Aeroelasticity, Fort Worth, Texas (November 6-7, 1958).

34. R.L. Bisplinghoff, H. Ashley, and R.L. Halfman. Aeroelasticity. Reading: Addison-Wesley Publishing Company, Inc., 1955.

35. W.P. Rodden, E.F. Farkas, and R.K. Oyama. "Subsonic and Sonic Aerodynamic Influence Coefficients from Unsteady Lifting Surface Theory; Analytical Development and Procedure for the IBM 7090 Computer." Norair Division, Northrop Corporation, Report NOR-61-56, 14 April 1961.

36. A.G. Webster. Partial Differential Equations of Mathematical Physics. New York: Dover Publications, 1955.

37. C. Lanczos. Applied Analysis. Englewood Cliffs: Prentice-Hall, Inc., 1956.

38. J.W. Miles. The Potential Theory of Unsteady Supersonic Flow. London: Cambridge University Press, 1959.

39. M.T. Landahl. Unsteady Transonic Flow. New York: Pergamon Press, 1961.

40. H. Liepmann and A. Roshko. Elements of Gasdynamics. New York: John Wiley and Sons Publishing Company, Inc., 1957.

41. M.T. Landahl, E.L. Mollo-Christensen, and H. Ashley. "Parametric Studies of Viscous and Nonviscous Unsteady Flows." OSR TR No. 55-13, 1955.

42. W. Eckhaus. "Two-Dimensional Transonic Unsteady Flow with Shock Waves." OSR TN No. 59-491; M.I.T. Fluid Dynamics Research Group Report No. 59-3, May 1959.

43. W.R. Sears. "Small Perturbation Theory." Princeton Aeronautical Paperbacks, No. 4, 1960, 42.

44. M.J. Lighthill. "On Sound Generated Aerodynamically -- I. General Theory." Proceedings of the Royal Society (London), A, 211 (1952), 564; "On Sound Generated Aerodynamically -- II. Turbulence as a Source of Sound." Proceedings of the Royal Society (London), A, 222 (1955), 1.

45. S. T. Chu and L.S.G. Kovasznay. "Nonlinear Interaction in a Viscous Heat Conducting Compressible Gas." Journal of Fluid Mechanics, 3 (1958), 494.

46. M.J. Lighthill. "The Flow Behind a Stationary Shock." Philosophical Magazine and Journal of Science, 40 (1949), 214.

47. J.W. Miles. "On Panel Flutter in the Presence of a Boundary Layer." Journal of the Aerospace Sciences, 26 (1959), 81.

48. H. Ashley. "A Preliminary Look at Sonic Aerodynamic Influence Coefficients," private communication, 1960.

49. W.P. Rodden and J.D. Revell. "Aerodynamic Influence Coefficients from Unsteady Slender-Wing Theory." North American Aviation, Inc., Report NA-57-1437, September 1957.

50. H. Merbt and M.T. Landahl. "Aerodynamic Forces on Oscillating Low-Aspect Ratio Wings in Compressible Flow." Kungliga Tekneska Hogskola Handlingar (Stockholm), Aero TN30 (1953).

51. R.W. Fralich and J.M. Hedgepeth. "Flutter Analysis of Rectangular Wings of Very Low-Aspect Ratio." NACA TN 4245, June 1958.

52. J.H. Cunningham, C.E. Watkins, and D.S. Woolston. "A Systematic Kernel Function Procedure for Determining Aerodynamic Forces on Oscillating or Steady Finite Wings with Subsonic Leading Edges and Supersonic Trailing Edges, and a Description of Coding the Procedure for an IBM 704 Computing Machine." NASA Memo, Langley Field, 1 December 1959.

53. C.E. Watkins and J.H. Berman. "On the Kernel Function of the Integral Equation Relating Lift and Downwash Distribution of Oscillating Wings in Supersonic Flow." NACA Report 1257, 1956.

54. S. Pines, J. Dugundji, and J. Neuringer. "Aerodynamic Flutter Derivatives for a Flexible Wing with Supersonic and Subsonic Edges." Journal of the Aeronautical Sciences, 22 (1955), 693.

55. G. Zartarian and P.T. Hsu. "Theoretical Studies on the Prediction of Unsteady Supersonic Airloads on Elastic Wings -- Part I. Investigations on the Use of Oscillatory Supersonic Aerodynamic Influence Coefficients." WADC TR 56-97, December 1955.

56. G. Zartarian. "Theoretical Studies on the Prediction of Unsteady Supersonic Airloads on Elastic Wings -- Part II. Rules for Application of Oscillatory Supersonic Aerodynamic Influence Coefficients." WADC TR 56-97, February 1956.

57. T.Y. Li. "Aerodynamic Influence Coefficients for an Oscillating Thin Wing in Supersonic Flow." Journal of the Aeronautical Sciences, 23 (1956), 613.

58. H. Ashley and L.V. Andrew. "Computation of Velocity Potential Aerodynamic Influence Coefficients for Unsteady Supersonic Flow-Application to Flutter Stability Calculations." Convair Division, General Dynamics Corporation, Memo DG-G141, 19 December 1956.

59. J.D. Revell, R.J. Carrington, D. Milligan, and T. Vivian. "Three-Dimensional, Supersonic, Unsteady Aerodynamic Influence Coefficients by an Extension of the NASA Kernel Function Procedure for Digital Computation." Northrop Corporation, NOR-61-105, 23 December 1961.

60. C.E. Watkins and J.H. Berman. "Velocity Potential and Air Forces Associated with a Triangular Wing in Supersonic Flow, with Subsonic Leading Edges, and Deforming According to a General Quadratic Equation." NACA TN 3009, 1953.

61. G.N. Lance. "The Lift of Twisted and Cambered Wings in Supersonic Flow." Aeronautical Quarterly, 6 (1955), 149.

62. W.P. Rodden and J.D. Revell. "Aerodynamic Influence Coefficients for the Delta Wing with a Subsonic Leading Edge in Supersonic Flow." North American Aviation, Inc., Report NA-59-72, 8 January 1959.

63. H. Ferguson and G. Guderly. "Extension of Evvard's Method to Oscillatory Supersonic Flow." Zhurnal Flugwissenschaften, 8, Heft 3 (1960), 76.

64. M.M. Stanišić. "On the Unsteady Motion of a Delta Wing in Supersonic Flight." Journal of the Aerospace Sciences, 27 (1960), 399.

65. W.P. Rodden, E.F. Farkas, P.E. Williams, and F.C. Slack. "Aerodynamic Influence Coefficients by Piston Theory: Analytical Development and Procedure for the IBM 7090 Computer." Northrop Corporation, Report NOR-61-57, 14 April 1961.

66. H.G. Morgan, V. Huckel, and H.L. Runyan. "Procedure for Calculating Flutter at High Supersonic Speed Including Camber Deflections, and Comparison with Experimental Results." NACA TN 4335, September 1958.

67. M. Van Dyke. "A Study of Second-Order Supersonic Flow Theory." NACA Report 1081, 1952.

68. H.G. Morgan, H.L. Runyan, and V. Huckel. "Theoretical Considerations of Flutter at High Mach Numbers." Journal of the Aerospace Sciences, 25 (1958), 371.

69. J.W. Miles. "Unsteady Flow at Hypersonic Speeds." Proceedings of the 11th Symposium of the Colston Research Society, Colston Papers, XI (April 6, 1959), 185.

70. W.D. Hayes and R.F. Probstein. Hypersonic Flow Theory. New York: Academic Press, 1959.

71. H. Kennet. "Estimation of Unsteady Airloads on Blunt Bodies in a Hypersonic Stream -- Part I. Theory." OSR TN 59-1273; M.I.T. Fluid Dynamics Research Group Report No. 59-6, October 1959.

72. M. Tobak and W. Wehrend. "Stability Derivatives of Cones at Supersonic Speeds." NACA TN 3788, September 1956.

73. C.M. Jackson. "A Semigraphical Method of Applying Impact Theory to an Arbitrary Body to Obtain the Hypersonic Aerodynamic Characteristics at an Angle of Attack and Side Slip." NASA TN D-795, May 1961.

74. L. Lees. "Hypersonic Flow." 5th International Aeronautical Conference, Los Angeles, California (1955), 241.

75. G. Zartarian, P.T. Hsu, and H. Ashley. "Dynamic Airloads and Aeroelastic Problems at Entry Mach Numbers." Journal of the Aerospace Sciences, 28 (1961), 209.

76. L. Lees. "Recent Developments in Hypersonic Flow." Jet Propulsion, 27 (1957), 1162.

77. H. Schlichting. Boundary Layer Theory. 4th ed. Trans. J. Kestin. New York: McGraw-Hill Book Company, Inc., 1960.

78. M. Tucker. "Approximate Calculation of Turbulent Boundary Layer Development in Compressible Flow." NACA TN 2337, 1951.

79. L. Lees. "Laminar Heat Transfer over Blunt Nosed Bodies at Hypersonic Flight Speeds." Jet Propulsion, 26 (1956), 257.

80. F.K. Moore. "On Local Flat Plate Similarity in the Hypersonic Boundary Layer." Journal of the Aerospace Sciences, 28 (1961), 753.

81. M. Bertram and T. Blackstock. "Some Simple Solutions to the Problem of Production of Boundary Layer Self-Induced Pressures." NASA TN D-798, April 1961.

82. R.J. Whalen. "Viscous and Inviscid Nonequilibrium Gas Flows." Presented at IAS 29th Annual Meeting, New York, January 1961; IAS Paper No. 61-23.

83. N.C. Freeman. "Nonequilibrium Flow of an Ideal Dissociating Gas." Journal of Fluid Mechanics, 4, Pt. 4 (1958), 407.

84. R.H. Smith and J.A. Menard. "Supercircular Entry and Recovery with Maneuverable Manned Vehicles." Aerospace Engineering, 20 (1961), 12.

85. M.J. Lighthill. "Supersonic Flow Past Slender Pointed Bodies of Revolution at Yaw." Quarterly Journal of Mechanics and Applied Mathematics, 1 (1948), 76.

86. M.D. Van Dyke. "First- and Second-Order Theory of Supersonic Flow Past Bodies of Revolution." Journal of Aeronautical Sciences, 18 (1951), 161.

87. J.D. Revell. "Second-Order Theory for Unsteady Supersonic Flow Past Slender Pointed Bodies of Revolution." Journal of the Aerospace Sciences, 27 (1960), 730.

88. W.H. Dorrance. "Nonsteady Flow about Pointed Bodies of Revolution." Journal of the Aeronautical Sciences, 18 (1951), 505.

89. J.W. Miles. "On Nonsteady Supersonic Flow about Pointed Bodies of Revolution." Journal of the Aeronautical Sciences, 19 (1952), 208.

90. H.S. Tsien. "Supersonic Flow over an Inclined Body of Revolution." Journal of the Aeronautical Sciences, 5 (1938), 480.

91. M.J. Lighthill. "Methods for Predicting Phenomena in the High-Speed Flow of Gases." Journal of the Aeronautical Sciences, 16 (1949), 69.

92. G.N. Ward. "Supersonic Flow Past Slender Pointed Bodies." Quarterly Journal of Mechanics and Applied Mathematics, 2 (1949), 75.

93. R. Bond and B. Packard. "Unsteady Aerodynamic Forces on a Slender Body of Revolution in Supersonic Flow." NASA TN D-859, May 1961.

94. W.P. Rodden. "The Flutter of Two-Dimensional Flat Panels with Equally Spaced Supports in a Supersonic Flow." Ph.D. dissertation, University of California, Los Angeles, 1 October 1957.

95. H.C. Nelson and H.J. Cunningham. "Theoretical Investigation of Flutter of Two-Dimensional Flat Panels with One Surface Exposed to Supersonic Potential Flow." NACA TN 3465, July 1955.

96. W.E. Milne. Numerical Calculus. Princeton: Princeton University Press, 1949.

97. M. Holt and S. Strack. "Supersonic Panel Flutter of a Cylindrical Shell of Finite Length." Journal of the Aerospace Sciences, 28 (1961), 197.

98. C.C. Lin. The Theory of Hydrodynamic Stability. London: Cambridge University Press, 1955.

99. W.R. Sears. "Some Aspects of Nonstationary Airfoil Theory and Its Practical Application." Journal of the Aeronautical Sciences, 8 (1941), 104.

100. J.D. Revell and W.P. Rodden. "A Rational Method for Utilizing Experimental Aerodynamic Data in Flutter Analyses Through the Use of Aerodynamic Influence Coefficient Matrices." North American Aviation, Inc., Report NA-59-867, 30 January 1959.

101. W.P. Rodden. "An Empirical Weighting Matrix for Use with Aerodynamic Influence Coefficients in Aeroelastic Analyses." Northrop Corporation, Report NOR-59-320, 1 April 1959.

ACKNOWLEDGEMENT

The authors wish to acknowledge Mr. Robert J. Carrington of Norair for his contribution to the discussions of harmonic gust and transient aerodynamic influence coefficients.

UNCLASSIFIED

analyses by modal methods is discussed. Several possibilities for the definition of transient AICs are suggested in a preliminary review of the transient problem.

The status of AICs is discussed for surfaces of various planforms by two-dimensional (i.e. strip) theories, and by three-dimensional theories, for slender bodies, and for panels, throughout the range of Mach number of current interest. A number of results are derived in detail to illustrate the technique of AIC development. An empirical weighting matrix that permits adjustment of the theoretical AICs to incorporate the results of experimental measurements is suggested, and, finally, a number of recommendations for further investigations are made.

UNCLASSIFIED

UNCLASSIFIED

analyses by modal methods is discussed. Several possibilities for the definition of transient AICs are suggested in a preliminary review of the transient problem.

The status of AICs is discussed for surfaces of various planforms by two-dimensional (i.e. strip) theories, and by three-dimensional theories, for slender bodies, and for panels, throughout the range of Mach number of current interest. A number of results are derived in detail to illustrate the technique of AIC development. An empirical weighting matrix that permits adjustment of the theoretical AICs to incorporate the results of experimental measurements is suggested, and, finally, a number of recommendations for further investigations are made.

UNCLASSIFIED

UNCLASSIFIED

analyses by modal methods is discussed. Several possibilities for the definition of transient AICs are suggested in a preliminary review of the transient problem.

The status of AICs is discussed for surfaces of various planforms by two-dimensional (i.e. strip) theories, and by three-dimensional theories, for slender bodies, and for panels, throughout the range of Mach number of current interest. A number of results are derived in detail to illustrate the technique of AIC development. An empirical weighting matrix that permits adjustment of the theoretical AICs to incorporate the results of experimental measurements is suggested, and, finally, a number of recommendations for further investigations are made.

UNCLASSIFIED

UNCLASSIFIED

analyses by modal methods is discussed. Several possibilities for the definition of transient AICs are suggested in a preliminary review of the transient problem.

The status of AICs is discussed for surfaces of various planforms by two-dimensional (i.e. strip) theories, and by three-dimensional theories, for slender bodies, and for panels, throughout the range of Mach number of current interest. A number of results are derived in detail to illustrate the technique of AIC development. An empirical weighting matrix that permits adjustment of the theoretical AICs to incorporate the results of experimental measurements is suggested, and, finally, a number of recommendations for further investigations are made.

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California. THE STATUS OF UNSTEADY AERODYNAMIC INFLUENCE COEFFICIENTS, by William P. Rodden, Aerospace Corporation, and James D. Revell, Northrop Corporation. 22 November 1961. [119]p. incl. illus. (Report No. TDR-930(2230-09)TN-2) (Contract AF 04(647)-930). Unclassified report

Unsteady aerodynamic influence coefficients (AICs) are defined that relate the force distribution to the downwash or deflection distribution throughout an aeroelastic system. Downwash is considered to arise from either motion or a gust, and the unsteadiness is considered to be oscillatory, transient, or static (a limit of the oscillatory case). Definitions are suggested for a set of oscillatory AICs for motion and for a gust, and for a set of static AICs. The use of AICs thus defined is illustrated in collocation formulations of several aeroelastic problems, and the use of the AICs in
(over)

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California. THE STATUS OF UNSTEADY AERODYNAMIC INFLUENCE COEFFICIENTS, by William P. Rodden, Aerospace Corporation, and James D. Revell, Northrop Corporation. 22 November 1961. [119]p. incl. illus. (Report No. TDR-930(2230-09)TN-2) (Contract AF 04(647)-930). Unclassified report

Unsteady aerodynamic influence coefficients (AICs) are defined that relate the force distribution to the downwash or deflection distribution throughout an aeroelastic system. Downwash is considered to arise from either motion or a gust, and the unsteadiness is considered to be oscillatory, transient, or static (a limit of the oscillatory case). Definitions are suggested for a set of oscillatory AICs for motion and for a gust, and for a set of static AICs. The use of AICs thus defined is illustrated in collocation formulations of several aeroelastic problems, and the use of the AICs in
(over)

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California. THE STATUS OF UNSTEADY AERODYNAMIC INFLUENCE COEFFICIENTS, by William P. Rodden, Aerospace Corporation, and James D. Revell, Northrop Corporation. 22 November 1961. [119]p. incl. illus. (Report No. TDR-930(2230-09)TN-2) (Contract AF 04(647)-930). Unclassified report

Unsteady aerodynamic influence coefficients (AICs) are defined that relate the force distribution to the downwash or deflection distribution throughout an aeroelastic system. Downwash is considered to arise from either motion or a gust, and the unsteadiness is considered to be oscillatory, transient, or static (a limit of the oscillatory case). Definitions are suggested for a set of oscillatory AICs for motion and for a gust, and for a set of static AICs. The use of AICs thus defined is illustrated in collocation formulations of several aeroelastic problems, and the use of the AICs in
(over)

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California. THE STATUS OF UNSTEADY AERODYNAMIC INFLUENCE COEFFICIENTS, by William P. Rodden, Aerospace Corporation, and James D. Revell, Northrop Corporation. 22 November 1961. [119]p. incl. illus. (Report No. TDR-930(2230-09)TN-2) (Contract AF 04(647)-930). Unclassified report

Unsteady aerodynamic influence coefficients (AICs) are defined that relate the force distribution to the downwash or deflection distribution throughout an aeroelastic system. Downwash is considered to arise from either motion or a gust, and the unsteadiness is considered to be oscillatory, transient, or static (a limit of the oscillatory case). Definitions are suggested for a set of oscillatory AICs for motion and for a gust, and for a set of static AICs. The use of AICs thus defined is illustrated in collocation formulations of several aeroelastic problems, and the use of the AICs in
(over)

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California. THE STATUS OF UNSTEADY AERODYNAMIC INFLUENCE COEFFICIENTS, by William P. Rodden, Aerospace Corporation, and James D. Revell, Northrop Corporation. 22 November 1961. [119]p. incl. illus. (Report No. TDR-930(2230-09)TN-2) (Contract AF 04(647)-930). Unclassified report

Unsteady aerodynamic influence coefficients (AICs) are defined that relate the force distribution to the downwash or deflection distribution throughout an aeroelastic system. Downwash is considered to arise from either motion or a gust, and the unsteadiness is considered to be oscillatory, transient, or static (a limit of the oscillatory case). Definitions are suggested for a set of oscillatory AICs for motion and for a gust, and for a set of static AICs. The use of AICs thus defined is illustrated in collocation formulations of several aeroelastic problems, and the use of the AICs in

(over)

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California. THE STATUS OF UNSTEADY AERODYNAMIC INFLUENCE COEFFICIENTS, by William P. Rodden, Aerospace Corporation, and James D. Revell, Northrop Corporation. 22 November 1961. [119]p. incl. illus. (Report No. TDR-930(2230-09)TN-2) (Contract AF 04(647)-930). Unclassified report

Unsteady aerodynamic influence coefficients (AICs) are defined that relate the force distribution to the downwash or deflection distribution throughout an aeroelastic system. Downwash is considered to arise from either motion or a gust, and the unsteadiness is considered to be oscillatory, transient, or static (a limit of the oscillatory case). Definitions are suggested for a set of oscillatory AICs for motion and for a gust, and for a set of static AICs. The use of AICs thus defined is illustrated in collocation formulations of several aeroelastic problems, and the use of the AICs in

(over)

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California. THE STATUS OF UNSTEADY AERODYNAMIC INFLUENCE COEFFICIENTS, by William P. Rodden, Aerospace Corporation, and James D. Revell, Northrop Corporation. 22 November 1961. [119]p. incl. illus. (Report No. TDR-930(2230-09)TN-2) (Contract AF 04(647)-930). Unclassified report

Unsteady aerodynamic influence coefficients (AICs) are defined that relate the force distribution to the downwash or deflection distribution throughout an aeroelastic system. Downwash is considered to arise from either motion or a gust, and the unsteadiness is considered to be oscillatory, transient, or static (a limit of the oscillatory case). Definitions are suggested for a set of oscillatory AICs for motion and for a gust, and for a set of static AICs. The use of AICs thus defined is illustrated in collocation formulations of several aeroelastic problems, and the use of the AICs in

(over)

UNCLASSIFIED

UNCLASSIFIED

Aerospace Corporation, El Segundo, California. THE STATUS OF UNSTEADY AERODYNAMIC INFLUENCE COEFFICIENTS, by William P. Rodden, Aerospace Corporation, and James D. Revell, Northrop Corporation. 22 November 1961. [119]p. incl. illus. (Report No. TDR-930(2230-09)TN-2) (Contract AF 04(647)-930). Unclassified report

Unsteady aerodynamic influence coefficients (AICs) are defined that relate the force distribution to the downwash or deflection distribution throughout an aeroelastic system. Downwash is considered to arise from either motion or a gust, and the unsteadiness is considered to be oscillatory, transient, or static (a limit of the oscillatory case). Definitions are suggested for a set of oscillatory AICs for motion and for a gust, and for a set of static AICs. The use of AICs thus defined is illustrated in collocation formulations of several aeroelastic problems, and the use of the AICs in

(over)

UNCLASSIFIED